Test-Efficiency Analysis of Random Self-Test of Sequential Circuits

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Abstract

In this paper we consider the problems left open in [6] and [8] in the area of test efficiency prediction for random pattern testing schemes proposed in the above papers. It is shown that, given a circuit with n primary inputs and the goal of maximizing expected pattern coverage, different pattern sampling distributions for its 2^n possible patterns can be partially ordered. The exact distributions for pattern coverage for both equiprobable and non-equiprobable pattern sampling distributions are derived. Approximations for pattern coverage distribution under equiprobable pattern sampling conditions and corresponding numerical results are presented. We also present numerical results on confidence levels for obtaining a specified pattern coverage. Finally, the distribution for the number R of test cycles required to achieve a specified pattern coverage is also derived. We derive and use the expression for the expected value of R to illustrate the increase in the effort of achieving a specified coverage j as j increases.

1 Introduction

Motivated by the work of Kim et al. [6] and Krasniewski and Pilarski [8], we address the problem of test efficiency in random testing of sequential circuits using BIST techniques. In [6] and [8] the authors describe two schemes in which the sequential circuit under test is first partitioned into a number of combinational blocks and, in test mode, random test vectors are applied to the different combinational blocks.

The number of distinct patterns applied to a logic module in r clock cycles (called the *pattern coverage* in r cycles) is considered as a measure of test efficiency in [6] and [8]. Both the papers ([6] and [8]) leave open a number of problems for which analytical solutions could not be obtained. In the following section we summarize the results presented in [6] and [8] and identify a set of problems that remain to be solved. In subsequent sections we present solutions to these problems.

2 Summary of [6] and [8]

The scheme proposed in [6] (called *Pattern Generator Driven* or PGD scheme) is illustrated in Figure 1. The circuit is partitioned into a collection of combinational logic (CL) blocks such that each CL block is separated from other blocks by registers. Those registers that are not at the output of any CL block are configured as complete *pseudo random pattern generators* (PRPG's) and all other registers that act as output registers for some CL block are configured as *multiple input signature registers* (MISR's). Some MISR's may also act as input registers for some CL blocks (such as R_2 in Fig. 1). The outputs of an MISR is a function of its state and its input. Thus the input pattern applied to block C_2 (in Figure 1) at time t is a function of the output of C_1 and the state of the MISR R_2 at time $(t-1)$. In this manner input test patterns to all blocks are driven by the PRPG's at the primary inputs of the circuit, hence the name PGD. Test results of each CL block are compacted by an MISR at its output. CL blocks that receive inputs directly from a PRPG are tested exhaustively. However, other blocks may or may not be tested exhaustively.

In the *Circular Self-Test Path* (CSTP) scheme proposed by Krasniewski and Pilarski [8], a set of registers are selected and connected as a linear feedback shift register (LFSR)

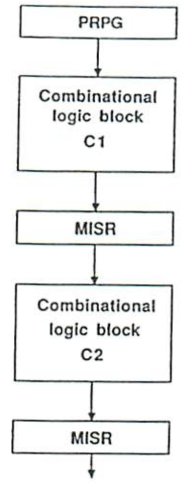

Figure 1: A system implementing PGD test technique.

implementing the polynomial $(1 + x^k)$ [4], where k is the number of FF's in the shift register chain. This LFSR generates random input patterns for different CL blocks in the circuit. Registers that are not included in the LFSR can be seen as pure delay elements. Figure 2 shows such a configuration.

Note that for both the PGD and the CSTP schemes cycling may occur if a previously visited state is repeated. This problem may be alleviated by allowing initiation of the FF's with different seeds in different test sessions.

As stated in [8], the ability of CSTP and PGD schemes to sensitize faults in a CL block depends on the number of distinct patterns that are applied to that block within a given number of clock cycles. This quantity is referred to as *pattern coverage.* In both [6] and [8] it is shown that as the number of clock cycles r tends to infinity, the probability of each bit of an LFSR being '1' or '0' approaches 0.5. This implies that all vectors that can be applied to a CL block are equally likely to occur. Under this assumption the authors of [6] and [8] derive an expression for the *expected pattern coverage* and use this as a measure for evaluating the fault detection capabilities of self-testing circuits. They indicate that obtaining the exact distribution of pattern coverage (even under simpler assumptions) is very difficult. For this reason estimates of error (i.e. variance) and asymptotic approximations to the exact distributions could not be obtained. Hence, extensive simulation results were presented. Furthermore, the problem of predicting the number of clock cycles required to

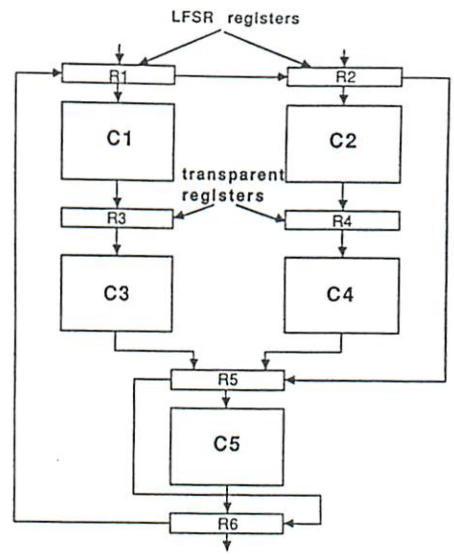

Figure 2: Implementation of CSTP test technique.

obtain a given pattern coverage has not been addressed in either paper. This is important in determining the duration for which a circuit should be in test mode in order to sufficiently test the circuit.

The objective of this paper is to present analytical solutions to the various problems that were left open in [6] and [8]. These problems are listed below.

Problem 1: Given a set of N possible test vectors $\{v_1, \ldots, v_N\}$, let $\tilde{p} = (p_1, \ldots, p_N)$ and $\tilde{q} = (q_1, \ldots, q_N)$ denote two pattern sampling distributions. That is p_i and q_i represent two probability assignments for selecting test vector v_i such that $\sum_{i=1}^{N} p_i = \sum_{i=1}^{N} q_i = 1$. We wish to derive a *sufficient* condition on $\tilde{p}$ and $\tilde{q}$ such that the expected pattern coverage under $\tilde{p}$ will be greater than the expected pattern coverage under $\tilde{q}$.

Problem 2: Given a CL block B with n inputs, let $C_{r,n}$ be a random variable that represents the number of distinct test vectors that are applied to B in r clock cycles. We wish to derive the distribution and moments of $C_{r,n}$ and obtain asymptotic approximations to the distribution when n and r are large. This is to be done for both the equiprobable and non-equiprobable pattern sampling distributions.

Problem 3: For a given pattern coverage α, let T_α denote the number of clock cycles needed to achieve a pattern coverage of α. We wish to derive the distribution and moments of T_α.

3 Notation

$(x)_j$	$x(x-1)(x-2)\dots(x-j+1)$
$\begin{pmatrix} x \\ y_1,\dots,y_k \end{pmatrix}$	$\frac{x!}{y_1!\,y_2!\cdots y_k!}$ such that $\sum_{i=1}^{k} y_i = x$.
$\Delta f(x)$	$f(x+1) - f(x)$
$\Delta^k(0^r)$	$\Delta^k(x^r)\Big\vert_{x=0}$
$\begin{bmatrix} r \\ k \end{bmatrix}$	Stirling's numbers of the first kind, i.e. $r!\begin{pmatrix} x \\ r \end{pmatrix} = \sum_k (-1)^{r-k} \begin{bmatrix} r \\ k \end{bmatrix} x^k$ (see [7]).
$\left\{ \begin{matrix} r \\ k \end{matrix} \right\}$	Stirling's numbers of the second kind $= \frac{\Delta^k(0^r)}{k!}$
$\mu_m(X)\ \left(\mu_{(m)}(X)\right)$	Ordinary (factorial) mth moments of a random variable X.
$FM_X(t)$	Factorial moment generating function of the random variable X
$E[X]$	Expected value of the random variable $X = \mu_1(X)$
$\sigma^2(X)$	Variance of a random variable X.
n	Number of inputs to a combinational logic block.
N	2^n

4 Problem 1: Partial Ordering of Pattern Sampling Distributions

Consider a CL block with n inputs. Let the set of all $N = 2^n$ possible binary input patterns of length n be denoted by $\{v_1, \ldots, v_N\}$. Let p_i denote the probability of selecting pattern v_i on any trial. Thus $\sum_{i=1}^{N} p_i = 1$. The vector $\tilde{p} = (p_1, \ldots, p_N)$ is called a *pattern sampling distribution.*

Let

$$X_{i,r} = \begin{cases} 1 & \text{if pattern } v_i \text{ is selected at least once in } r \text{ clock cycles} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Then the pattern coverage in r clock cycles, $C_{r,n}$ is given by

$$C_{r,n} = \sum_{i=1}^{N} X_{i,r} \tag{2}$$

and the expected value of $C_{r,n}$ expressed as a function of the pattern sampling distribution $\tilde{p}$ denoted by $\phi(\tilde{p})$, is given by

$$\phi(\tilde{p}) = E(C_{r,n}) = \sum_{i=1}^{N} E(X_{i,r}) = \sum_{i=1}^{N} \left(1 - (1 - p_i)^r\right). \tag{3}$$

Given two pattern sampling distributions $\tilde{p}$ and $\tilde{q}$, we wish to find a relation between $\tilde{p}$ and $\tilde{q}$ such that $\phi(\tilde{p}) \geq \phi(\tilde{q})$. In order to establish such a relation we need to define *majorization* and use a well known theorem regarding Schur concave functions [9].

Definition 1: (Majorization) (see [9]) For a given vector $\tilde{p}$, let $[\tilde{p}]$ be a vector with the elements of $\tilde{p}$ arranged in decreasing order, i.e. if $\tilde{p} = (p_1, \ldots, p_N)$ then $[\tilde{p}] = (p_{[1]}, \ldots, p_{[N]})$ where $p_{[1]} \geq \cdots \geq p_{[N]}$. A vector $\tilde{p}$ is said to be *majorized* by a vector $\tilde{q}$ (denoted by $\tilde{p} \prec \tilde{q}$) if

$$\sum_{i=1}^{k} p_{[i]} \leq \sum_{i=1}^{k} q_{[i]} \qquad 1 \leq k \leq N$$

and

$$\sum_{i=1}^{N} p_{[i]} = \sum_{i=1}^{N} q_{[i]}. \tag{4}$$

□

Theorem 1: Let F be a real valued continuous function on $A \subset \Re^n$, and be continuously differentiable on the interior of A. Then for any two vectors $x, y \in A$, $x \prec y$ implies $F(x) \geq F(y)$ *if and only if* $\partial F([z])/\partial z_{[k]}$ is increasing in k for all $z \in A$.

Proof: See [9]. □

Lemma 1: Given two pattern sampling distributions $\tilde{p}$ and $\tilde{q}$, if $\tilde{p} \prec \tilde{q}$ then $\phi(\tilde{p}) \geq \phi(\tilde{q})$.

Proof: From the definition of $\phi(\tilde{p})$ given in Equation (3), we see that $\phi(\tilde{p}) = \phi([\tilde{p}])$, and

$$\frac{\partial \phi([\tilde{p}])}{\partial p_{[k]}} = r(1 - p_{[k]})^{r-1}. \tag{5}$$

Since $p_{[k]} \geq p_{[k+1]}$, we see that

$$\frac{\partial \phi([\tilde{p}])}{\partial p_{[k]}} \leq \frac{\partial \phi([\tilde{p}])}{\partial p_{[k+1]}}. \tag{6}$$

Hence by theorem 1, we conclude that $\phi(\tilde{p}) \geq \phi(\tilde{q})$ whenever $\tilde{p} \prec \tilde{q}$. □

Lemma 1 provides a simple criterion for determining which one of two pattern sampling distributions related by the relation "$\prec$" yields a larger expected pattern coverage. We now show that the expected pattern coverage is maximized when all test patterns are equally likely to be sampled in any trial.

Corollary 1: The expected pattern coverage $\phi(\tilde{p})$ given by Equation (3) is maximum when $p_i = 1/N$, $1 \leq i \leq N$.

Proof: It suffices to show that $(\frac{1}{N}, \ldots, \frac{1}{N}) \prec \tilde{q}$ for all $\tilde{q} = (q_1, \ldots, q_N) \in [0,1]^N$ such that $\sum_{i=1}^{N} q_i = 1$, or equivalently

$$\frac{k}{N} \leq \sum_{i=1}^{k} q_{[i]} \qquad 1 \leq k \leq N. \tag{7}$$

Clearly $1/N \leq q_{[1]}$, since otherwise $\sum_{i=1}^{N} q_{[i]} < 1$. Hence inequality (7) is true for $k = 1$. Assume that this inequality holds for $k = j$ and suppose that it fails for $k = j + 1$. That is

$$\frac{j}{N} \leq \sum_{i=1}^{j} q_{[i]} \tag{8}$$

and

$$\frac{j+1}{N} > \sum_{i=1}^{j+1} q_{[i]}. \tag{9}$$

Condition (9) given (8) implies that $1/N > q_{[j+1]}$. From condition (9) we have

$$1 - \frac{j+1}{N} = \frac{N-j-1}{N} < 1 - \sum_{i=1}^{j+1} q_{[i]} = \sum_{i=j+2}^{N} q_{[i]}. \tag{10}$$

Since $q_{[N]} \leq \cdots \leq q_{[j+1]}$, we have

$$\sum_{i=j+2}^{N} q_{[i]} \leq (N-j-1)q_{[j+1]} < \frac{N-j-1}{N}. \tag{11}$$

Conditions (10) and (11) lead to the contradiction

$$\frac{N-j-1}{N} < \frac{N-j-1}{N}.$$

Therefore if the inequality (7) holds for $k = j$ then it must hold for $k = j+1$. This along with the fact that the inequality holds for $k = 1$ proves that the inequality (7) holds for all k, $1 \leq k \leq N$. □

5 Problem 2: Distribution of Pattern Coverage

In this section we derive the exact distribution and moments of pattern coverage as a function of the duration of test. We examine two cases, namely

1) equiprobable pattern sampling distribution, and

2) non-equiprobable pattern sampling distribution.

As we shall see, the exact distributions are difficult to evaluate numerically when their parameters are large. For this reason we present useful asymptotic approximations to the exact distributions and corresponding numerical results.

5.1 Equiprobable Pattern Sampling Distribution

Consider a CL block with n primary inputs and assume that each of the $N = 2^n$ possible test patterns is equally likely to be applied to its inputs in any clock cycle. Suppose testing has been in progress for some time and i distinct test patterns out of N have been applied to the CL block. Let $P_{i,j,r}$ denote the probability that $(j - i)$ additional distinct test patterns will be applied to the CL block in r additional clock cycles given that i distinct patterns have already been applied. Thus, using the above notation, the probability that pattern coverage in r clock cycles is j is given by

$$P\{C_{r,n} = j\} = P_{0,j,r} \tag{12}$$

Theorem 2: Under equiprobable pattern sampling assumption

$$P_{i,j,r} = \binom{N-i}{j-i} \frac{(-1)^i (j-i)!}{N^r} \sum_{k=0}^{i} (-1)^k \left[\begin{matrix} i \\ k \end{matrix} \right] \left\{ \begin{matrix} k+r \\ j \end{matrix} \right\} \qquad i \le j \le i + r \tag{13}$$

Proof: The recurrence relation of $P_{i,j,r}$ is given by

$$P_{i,j,r} = \frac{N - j + 1}{N} P_{i,j-1,r-1} + \frac{j}{N} P_{i,j,r-1} \tag{14}$$

with $P_{i,i,r} = (i/N)^r$. Let $g_{i,j}(y)$ denote the generating function of $P_{i,j,r}$ with respect to r. That is

$$g_{i,j}(y) = \sum_{r=0}^{\infty} P_{i,j,r}\, y^r. \tag{15}$$

Multiplying both sides of Equation (14) by y^r and summing over r we obtain

$$g_{i,j}(y) = \frac{(N - j + 1)y}{N - jy}\, g_{i,j-1}(y) \tag{16}$$

with

$$g_{i,i}(y) = \frac{N}{N - iy}. \tag{17}$$

Expanding Equation (16) we obtain

$$g_{i,j}(y) = \frac{(N-i)_{(j-i)}}{\prod_{k=i}^{j} \left(1 - \frac{ky}{N}\right)} \left(\frac{y}{N}\right)^{j-i} = \frac{(y/N)^j}{\prod_{k=1}^{j} \left(1 - \frac{ky}{N}\right)} \frac{\prod_{k=1}^{i-1} \left(1 - \frac{ky}{N}\right)}{(y/N)^i} (N-i)_{(j-i)}. \tag{18}$$

Using the following identities [7]

$$\frac{z^j}{\prod_{m=1}^{j}(1-mz)} = \sum_{t=j}^{\infty} \left\{ \begin{array}{c} t \\ j \end{array} \right\} z^t$$

and

$$\frac{\prod_{k=1}^{i-1}\left(1-\frac{ky}{N}\right)}{(y/N)^i} = \left(\frac{N}{y}\right)_i = \sum_{k=0}^{i}(-1)^{i-k} \left[\begin{array}{c} i \\ k \end{array} \right] \left(\frac{N}{y}\right)^k$$

we obtain

$$g_{i,j}(y) = (-1)^i\,(N-i)_{(j-i)} \sum_{m=j}^{\infty}\sum_{k=0}^{i} \left\{ \begin{array}{c} m \\ j \end{array} \right\} \left[\begin{array}{c} i \\ k \end{array} \right] (-1)^k \frac{y^{m-k}}{N^{m-k}}. \tag{19}$$

$P_{i,j,r}$ is the coefficient of y^r in $g_{i,j}(y)$. Setting $m = k + r$ in Equation (19) we obtain

$$P_{i,j,r} = \left(\begin{array}{c} N-i \\ j-i \end{array} \right) \frac{(-1)^i(j-i)!}{N^r} \sum_{k=0}^{i}(-1)^k \left[\begin{array}{c} i \\ k \end{array} \right] \left\{ \begin{array}{c} k+r \\ j \end{array} \right\}. \tag{20}$$

□

From Equations (12) and (20) we obtain the probability that the pattern coverage in r clock cycles equals j to be

$$P\{C_{r,n} = j\} = P_{0,j,r} = \frac{j!}{N^r} \left(\begin{array}{c} N \\ j \end{array} \right) \left\{ \begin{array}{c} r \\ j \end{array} \right\} = \frac{(N)_j}{N^r} \frac{\Delta^j(0^r)}{j!}. \tag{21}$$

Note: From Equation (21) it is easy to obtain the probability that no test pattern is repeated in r clock cycles. This is given by

$$P_{0,r,r} = \frac{(N)_r}{N^r} = \prod_{k=0}^{r-1}\left(1-\frac{k}{N}\right). \tag{22}$$

Equation (22) is similar to the one derived in [8] but with different limits on the index k. This is because the authors of [8] computed the probability of sampling a new test pattern in the first clock cycle as $(1-1/N)$. This is incorrect. The correct probability is 1 since in the first clock cycle we are sure to sample a new pattern.

5.1.1 Moments of $C_{r,n}$

In this section we derive expressions for the moments of $C_{r,n}$. This will lead to suitable approximations for $P\{C_{r,n} = j\}$. Equation (21) shows that $P\{C_{r,n} = j\}$ can be expressed in terms of powers of *differences of zero.* For this reason the simplest way to derive the moments of $C_{r,n}$ is to first obtain the factorial moments.

Lemma 2: The kth factorial moment of $C_{r,n}$, denoted by $\mu_{(k)}(r,n)$, is given by

$$\mu_{(k)}(r,n) = E\left[(C_{r,n})_k\right] = \frac{(N)_k}{N^r}\Delta^k\left[(N-k)^r\right]. \tag{23}$$

Proof: The factorial moment generating function of $C_{r,n}$, denoted by $FM_C(t)$, is given by

$$\begin{aligned} FM_C(t) &= E\left[(1+t)^{C_{r,n}}\right] = \frac{1}{N^r}\sum_{j=0}^{N}(1+t)^j(N)_j\frac{\Delta^j(0^r)}{j!} \\ &= \frac{1}{N^r}(1+\Delta+t\Delta)^N(0^r) \\ &= \frac{1}{N^r}\sum_{k=0}^{N}\binom{N}{k}t^k\Delta^k(1+\Delta)^{N-k}(0^r). \end{aligned} \tag{24}$$

We can express $(1+\Delta)^m f(x)$ as $f(x+m)$. Therefore $(1+\Delta)^m(x^r) = (x+m)^r$ and $(1+\Delta)^m(0^r) = m^r$. Hence

$$FM_C(t) = \frac{1}{N^r}\sum_{k=0}^{N}(N)_k\Delta^k\left[(N-k)^r\right]\frac{t^k}{k!} \tag{25}$$

and $\mu_{(k)}(r,n)$ is the coefficient of $\frac{t^k}{k!}$ in the expansion of $FM_C(t)$. □

From the definition of factorial moments we can see that the expected value of $C_{r,n}$, denoted by $\mu_1(r,n)$, is given by

$$\mu_1(r,n) = \mu_{(1)}(r,n) = E(C_{r,n}) \tag{26}$$

and the variance of $C_{r,n}$, denoted by $\sigma^2(r,n)$, can be expressed as

$$\sigma^2(r,n) = \mu_{(2)}(r,n) + \mu_{(1)}(r,n) - \mu_{(1)}^2(r,n). \tag{27}$$

From Equation (23) the first factorial moment is given by

$$\mu_{(1)}(r,n) = \frac{N}{N^r}\left(N^r - (N-1)^r\right) = N\left(1 - \left(1 - \frac{1}{N}\right)^r\right). \tag{28}$$

Equation (28) gives the expected pattern coverage. The variance $\sigma^2(r,n)$ is obtained as follows.

$$\mu_{(2)}(r,n) = N(N-1)\left[1 - 2\left(1 - \frac{1}{N}\right)^r + \left(1 - \frac{2}{N}\right)^r\right] \tag{29}$$

Substituting Equation (28) and (29) into Equation (27), we obtain

$$\sigma^2(r,n) = N\left[\left(1 - \frac{1}{N}\right)^r + (N-1)\left(1 - \frac{2}{N}\right)^r - N\left(1 - \frac{1}{N}\right)^{2r}\right]. \tag{30}$$

From Equation (28) it is clear that the expected pattern coverage tends to N as r tends to infinity. However, the variance, $\sigma^2(r,n)$ is maximum when $r = N$ and tends to zero as $r \to \infty$ for fixed N.

5.1.2 Approximations to $P\{C_{r,n} = j\}$

The $p.d.f$ of $C_{r,n}$ given by Equation (21) is difficult to evaluate numerically when r and N are large. To obtain suitable approximations to $P\{C_{r,n} = j\}$ we can either approximate $\Delta^j(0^r)$ or determine the limiting $p.d.f.$ as r and N both tend to infinity.

The problem of approximating differences of zero has been around since the 18th century. Since then a number of approximations have been derived. However, the approximations due to Laplace [2], are by far, the most accurate and probably cannot be improved upon. Laplace's various approximations [2] are quite general in that they approximate $\Delta^j(x^r)$. For our purposes, we use Laplace's approximation to $\Delta^j(0^r)$ which is stated below.

Let $x = a \neq 0$ be a solution to the equation

$$x = \frac{r+1}{j}\left(1 - e^{-x}\right). \tag{31}$$

Then Laplace's approximation states that

$$\Delta^j(0^r) \approx \frac{\left(\frac{r}{a}\right)^{r+1} e^{-r}\left(e^a - 1\right)^j}{\left(\frac{r(r+1)}{a^2} - rj\frac{e^a}{(e^a-1)^2}\right)^{1/2}}\left\{1 + \frac{15l'^2 - 12ll''}{16l^3} + \cdots\right\} \tag{32}$$

where

$$
\begin{aligned}
l &= -\frac{r+1}{2a^2} - \frac{j}{2}\left(\frac{e^a}{e^a-1}\right) + \frac{j}{2}\left(\frac{e^a}{e^a-1}\right)^2 \\
l' &= -\frac{r+1}{3a^3} + \frac{j}{6}\left(\frac{e^a}{e^a-1}\right) - \frac{j}{2}\left(\frac{e^a}{e^a-1}\right)^2 + \frac{j}{3}\left(\frac{e^a}{e^a-1}\right)^3 \\
l'' &= -\frac{r+1}{4a^4} + \frac{j}{24}\left(\frac{e^a}{e^a-1}\right) + \frac{7j}{24}\left(\frac{e^a}{e^a-1}\right)^2 - \frac{j}{2}\left(\frac{e^a}{e^a-1}\right)^3 + \frac{j}{4}\left(\frac{e^a}{e^a-1}\right)^4
\end{aligned}
$$

In our computation we retain only the first two terms in the series expansion given in Equation (32). Substituting the RHS of Equation (32) for $\frac{\Delta^j(0^r)}{j!}$ in Equation (21) yields a simple formula for computing $P\{C_{r,n} = j\}$. Note that the above approximation requires a solution to Equation (31). This is quite easily obtained by Newton's method for finding roots. Alternatively, tabulated solutions to Equation (31) may be found in [1].

We now show that the *p.d.f.* of $C_{r,n}$ can be approximated by a binomial density function when r and N both tend to infinity in such a way that r/N remains constant. This is most easily done by considering the factorial moments of $C_{r,n}$. Let $r \to \infty$ and $N \to \infty$ with $r/N = \lambda$. The kth order factorial moment, given by Equation (23) can be expressed as

$$\mu_{(k)}(r,n) = \frac{(N)_k}{N^r}\Delta^k\left[(N-k)^r\right] = (N)_k \sum_{i=0}^{k}(-1)^i \binom{k}{i}\left(1-\frac{i}{N}\right)^r. \tag{33}$$

When N is large, $\ln(1-i/N) \approx -i/N$. Therefore $(1-i/N)^r \approx e^{-\lambda i}$. Hence

$$\mu_{(k)}(r,n) \approx (N)_k \sum_{i=0}^{k}\binom{k}{i}\left(-e^{-\lambda}\right)^i = (N)_k\left(1-e^{-\lambda}\right)^k. \tag{34}$$

The kth factorial moment of a binomial random variable with parameters N and p is $p^k\,(N)_k$. Therefore Equation (34) shows that the *p.d.f.* of $C_{r,n}$ may be approximated by a binomial distribution with parameters N and $\left(1-e^{-\lambda}\right)$ when N and r are large with $r/N = \lambda$.

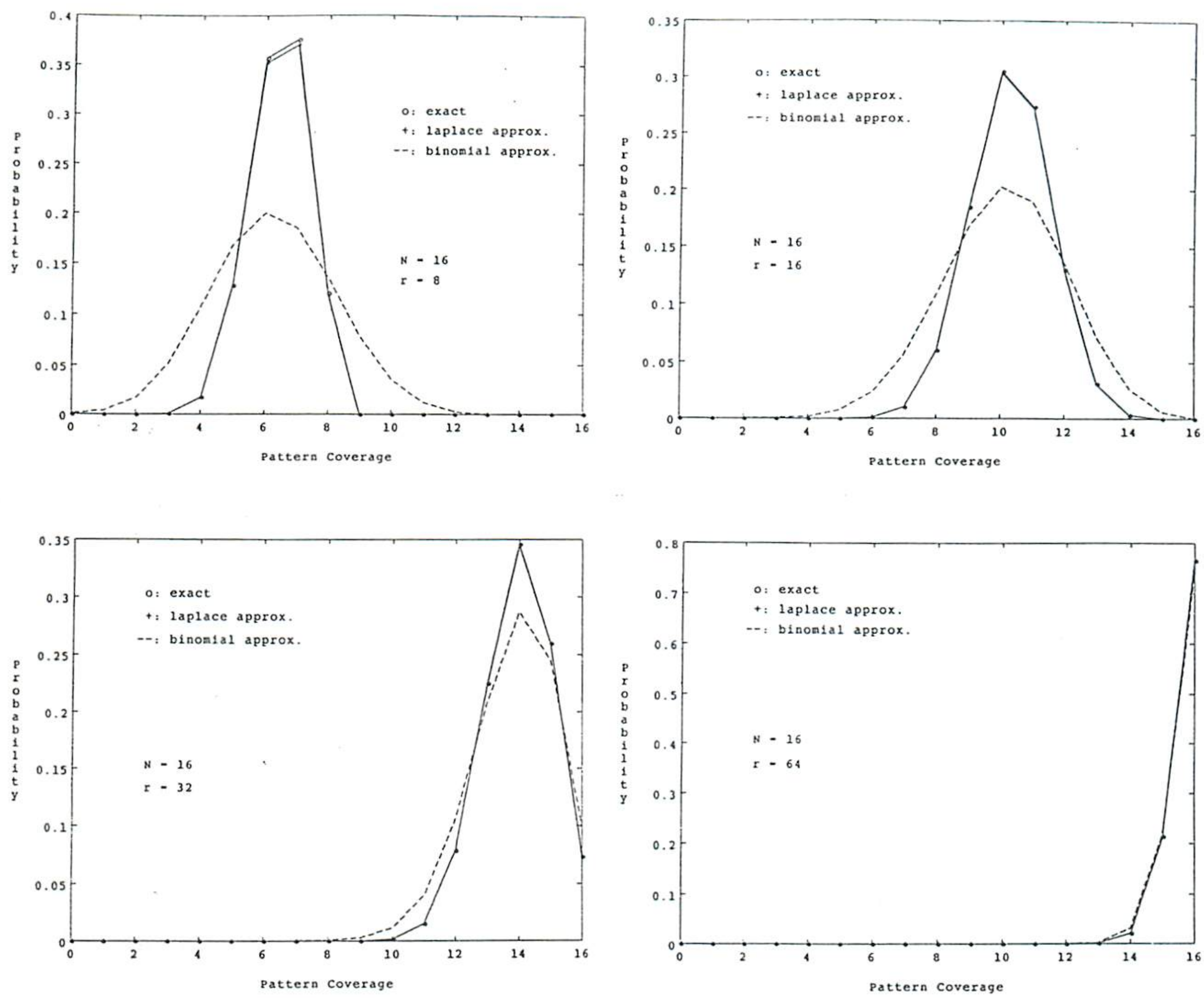

Figure 3: Probability mass function of $C_{r,n}$ for $N = 16$.

5.1.3 Numerical Results

In this section we present a comparison between the exact distribution function of $C_{r,n}$ and its approximations proposed in the previous section for different n and r. Figures 3, 4, 5 and 6 show exact values of $P\{C_{r,n} = j\}$ and its approximations for different values of N. It can be seen that Laplace's approximations for Stirling's numbers are excellent and the probability computations using these approximate values yield numbers that are quite close to the exact ones. Further, one can note that the binomial approximations to the *pmf* increase in accuracy as r increases and therefore can be used for computations instead of the exact probability expressions.

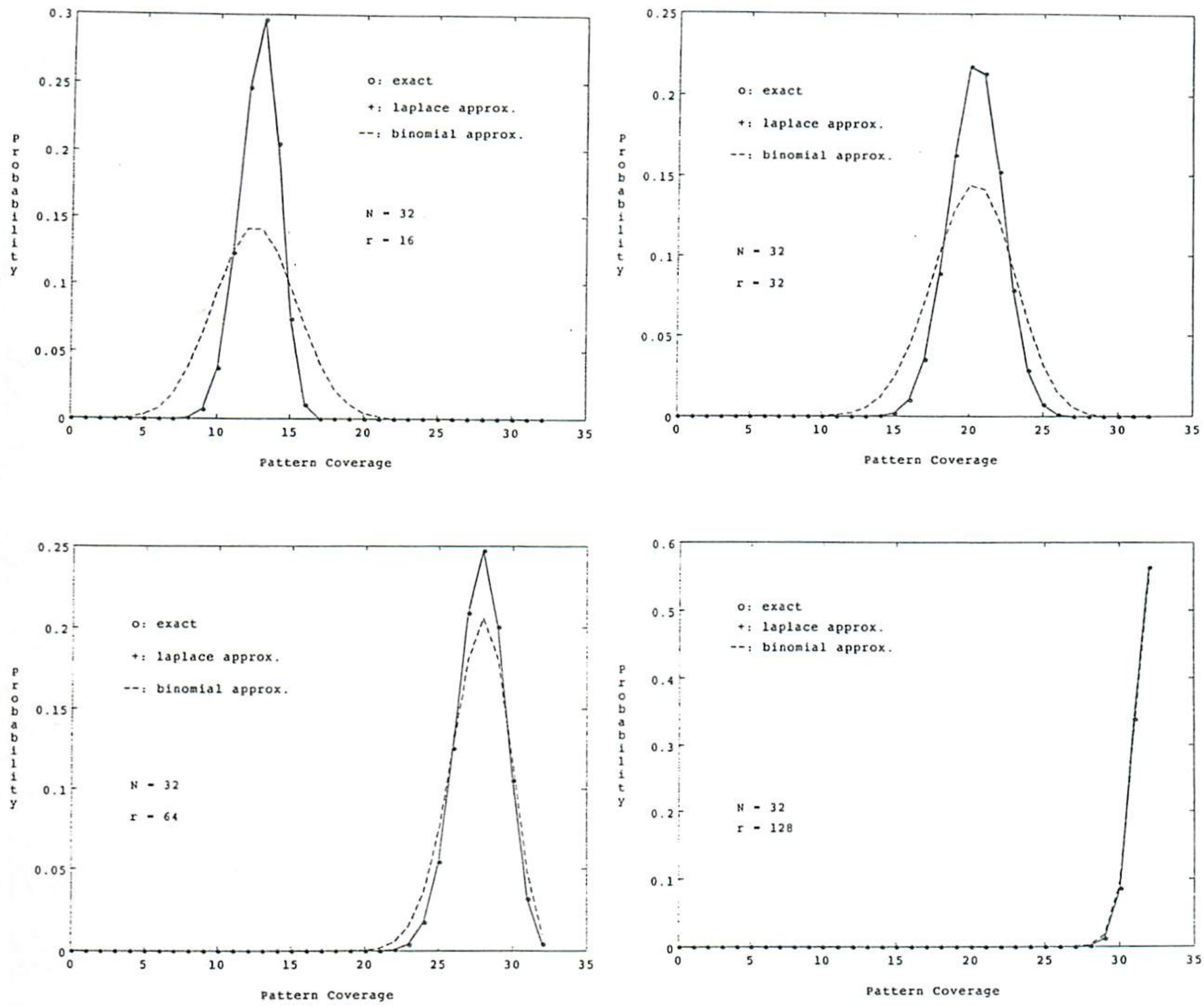

Figure 4: Probability mass function of $C_{r,n}$ for $N = 32$.

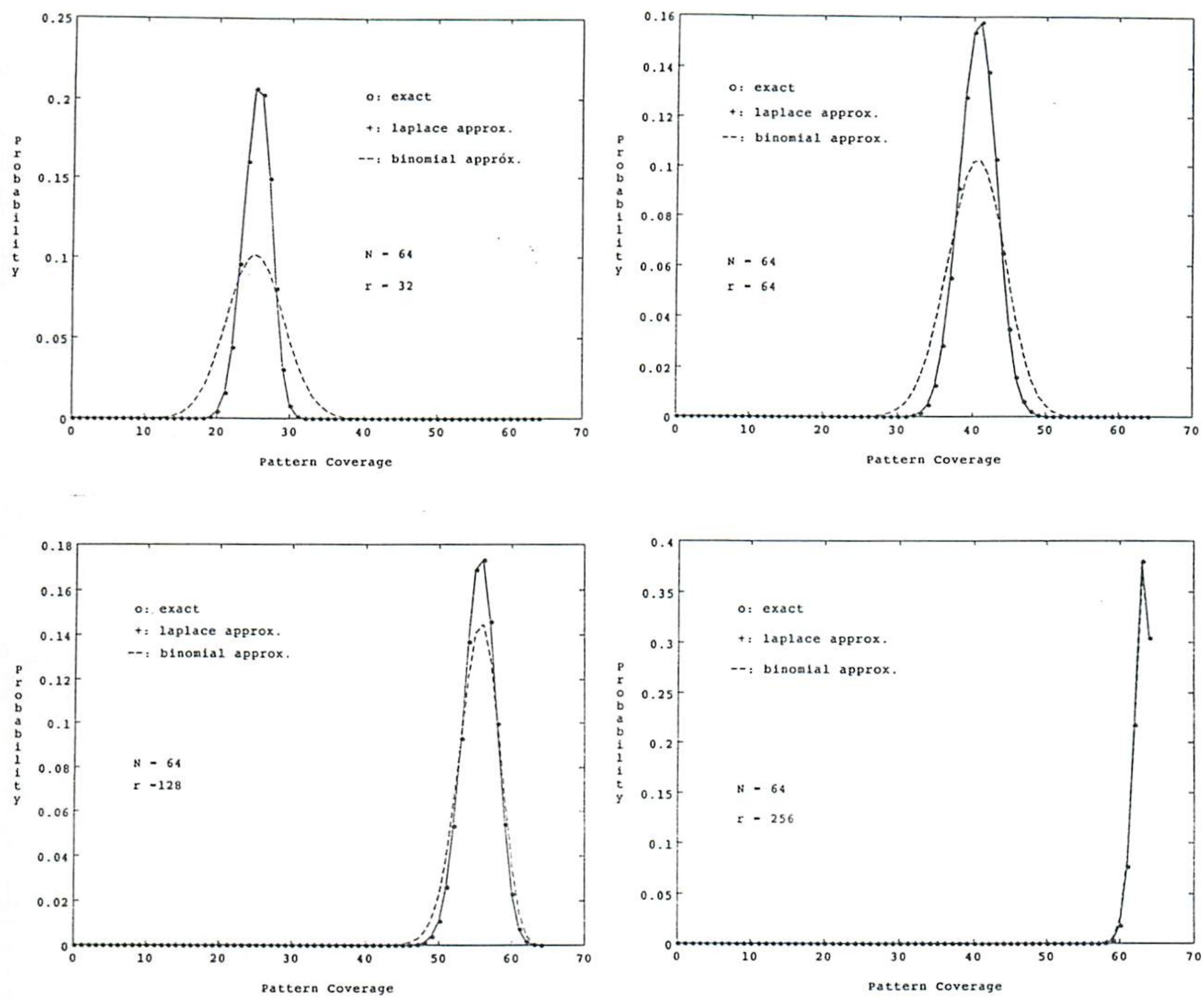

Figure 5: Probability mass function of $C_{r,n}$ for $N = 64$.

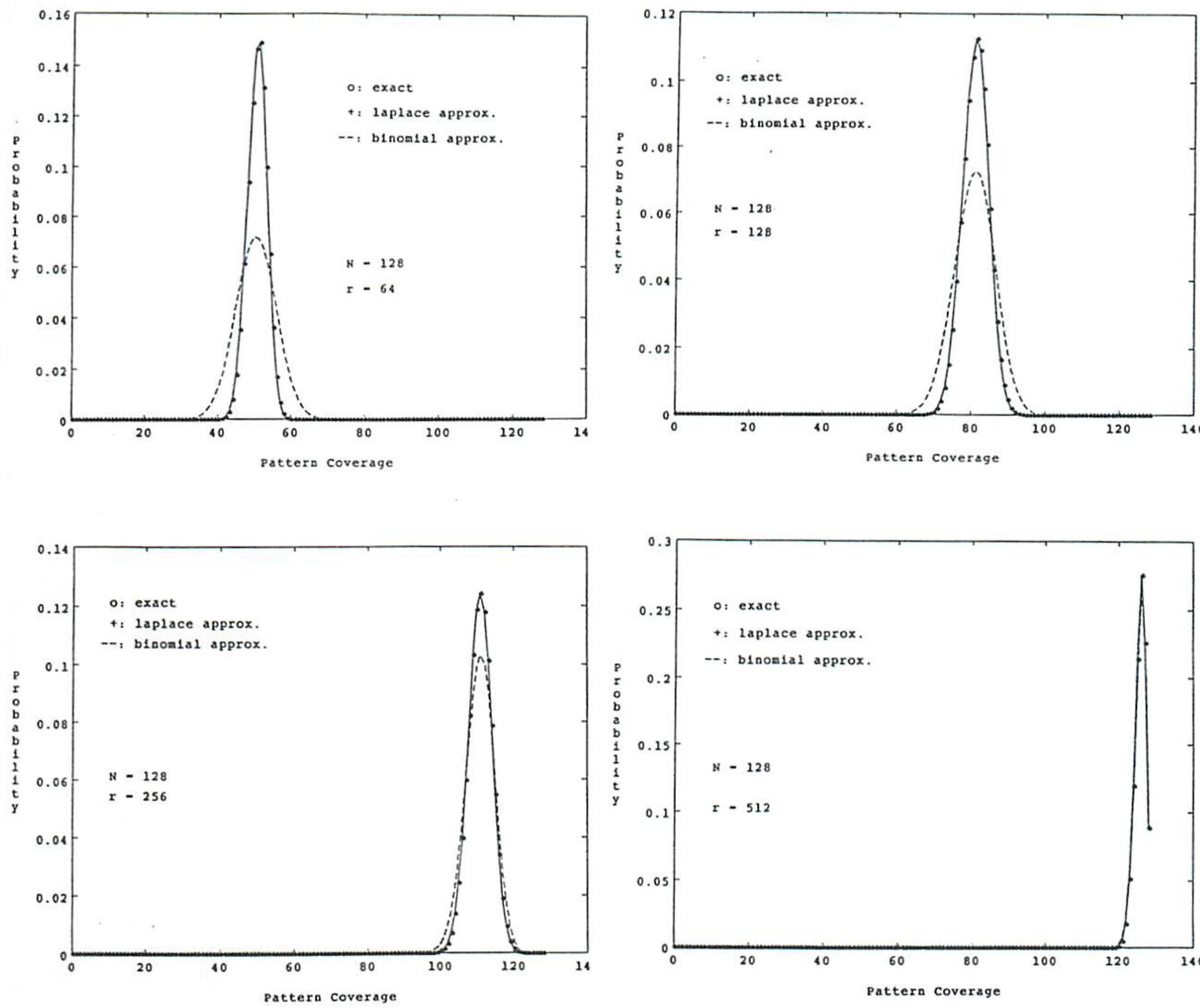

Figure 6: Probability mass function of $C_{r,n}$ for $N = 128$.

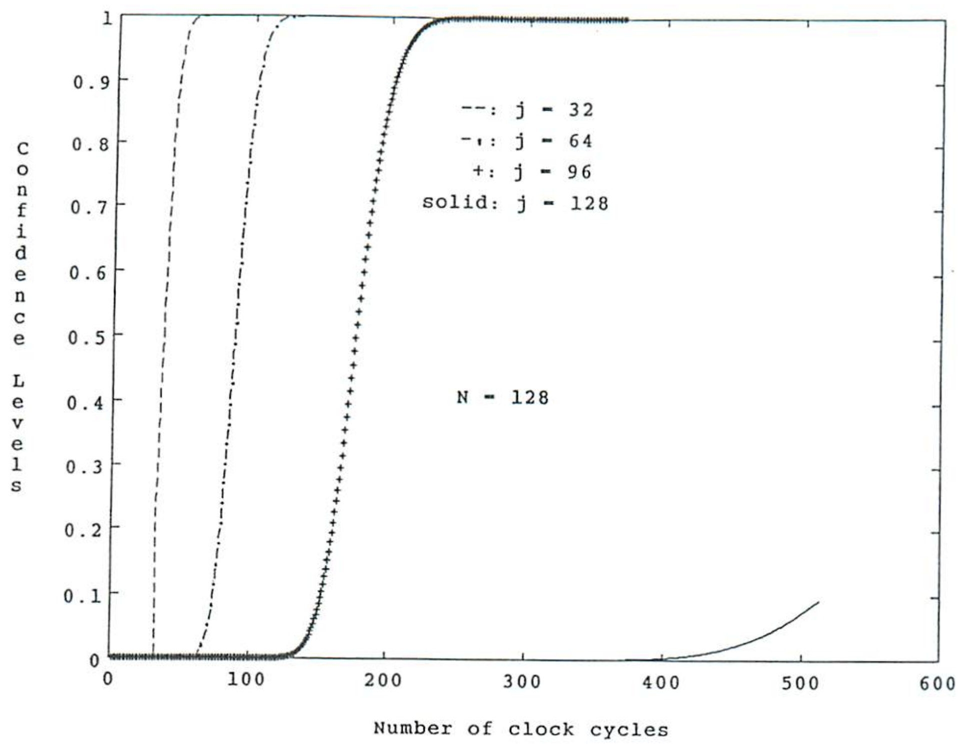

Figure 7: Confidence of achieving a coverage of j or more (out of N) patterns for differnt values of r, the number of test cycles used.

Our interest is also in determining the probability of obtaining a pattern coverage of at least j for different values of r. Thus,

$$P\{C_{r,n} \geq j \mid N, r\} = \sum_{k=j}^{N} P_{0,k,r} = CFD_{j,r}$$

where $CFD_{j,r}$ denotes the confidence of obtaining a pattern coverage of at least j in r clock cycles. In the following computations we use the binomial approximation derived above. Confidence levels for a circuit with $n = 7$ primary inputs with varying r and four different values of j are shown in Figure 7. Similar confidence values can be computed easily for other values of n. Let $r_{\alpha,j}$ denote the value of r for which the confidence of obtaining a coverage greater than a specified j is greater than α. It is interesting to note that the ratio $\frac{r_{\alpha,j}}{j}$ increases with j. This simply means that the difficulty of obtaining a coverage of j or more increases with j.

5.2 Non-Equiprobable Pattern Sampling Distribution

In the CSTP technique, as was mentioned earlier, there may be some combinational logic modules that get their inputs from pure delay registers, e.g. block $C3$ in Fig. 2. Since input registers to such blocks are not part of the LFSR, equiprobability of sampling input patterns to those CL blocks cannot be guaranteed. The only reasonable assumption that we can make about the sampling distribution of these input patterns is that there exists a limiting distribution which is most likely non-uniform. Therefore, in order to predict pattern coverage of these blocks we need to model the process of sampling input patterns differently.

Further, although it has been proved that in both PGD and CSTP schemes the limiting sampling distribution is the equiprobable one, there is no way of determining the initial sampling distribution. Moreover, it is believed to be fair to assume that the initial distribution is not equiprobable. The time required for the sampling distribution to settle down to within a $\pm\alpha$ tolerance of the limiting distribution increases as α decreases (see [8]). Therefore, depending on the value of α the *settling time* may be quite large. Due to this reason, the distribution of Y derived in section 3.1 may not be valid until the sampling distribution reaches tolerable limits.

While the sampling distribution is in the process of settling down, the distribution of pattern coverage can be expressed as a function of a non-equiprobable sampling distribution. In general the actual sampling distribution at any time during the settling process is not available. Further, the sampling distribution is changing in each clock cycle. In order to model this dynamic process, for the purpose of predicting pattern coverage we *simulate* an average behaviour of the sampling distribution. This model and results derived for it are described below.

Let N be the number of possible patterns for a circuit. Let the sampling distribution be denoted by the vector $\tilde{p} = (p_1, \ldots, p_N)$, i.e. p_i is the probability of selecting pattern v_i in a random trial. We had proved in section 4 that *distribution vectors* such as $\tilde{p}$ can be partially ordered with respect to their average ability to capture input patterns. In this section we present the actual distribution function for the number of patterns covered in r test cycles.

Let $C_{r,n}$ denote the pattern coverage in r test cycles. Let r_i denote the number of times pattern v_i was selected in r clock cycles, i.e. $\sum_{i=1}^{N} r_i = r$. Defining the set W_r as the set of vectors $(r_1, \ldots, r_N)$ such that $\sum_{i=1}^{N} r_i = r$, let $S_j \subset W_r$ be such that if $(r_1, \ldots, r_N) \in S_j$ then exactly j out of N components (r_i's) are greater than zero. Then the probability of $C_{r,n}$ being j is given by

$$P\{C_{r,n} = j \mid N, r\} = \sum_{s \in S_j} \binom{r}{s_1, \ldots, s_N} \prod_{i=1}^{N} (p_i)^{s_i} \tag{35}$$

where $s = (s_1, \ldots, s_N)$. Although equation (35) is difficult to evaluate, we can compute the expected value and variance of $C_{r,n}$ quite easily from N, r and $\tilde{p}$. Expected pattern coverage has been derived in section 4 and given in equation (3) as

$$E[C_{r,n}] = \sum_{i=1}^{N} \left(1 - (1 - p_i)^r\right).$$

From this we can get values for the average pattern coverage given N, r and $\tilde{p}$.

To obtain the variance of $C_{r,n}$ we need to use definition 2 of section 4. Let $X_{i,r}$ be as in definition 2. We define $\overline{X}_{i,r}$ as the indicator random variable such that $\overline{X}_{i,r} = 1 - X_{i,r}$. We know that $C_{r,n} = \sum_{i=1}^{N} X_{i,r}$. We can now prove the following proposition.

Proposition 1:

$$Var(C_{r,n}) = \sum_{i=1}^{N} Var(\overline{X}_{i,r}) + 2 \sum_{i} \sum_{<j} Cov(\overline{X}_{i,r}, \overline{X}_{j,r}) \tag{36}$$

Proof: We know that

$$Var(C_{r,n}) = \sum_{i=1}^{N} Var(X_{i,r}) + 2 \sum_{i} \sum_{<j} Cov(X_{i,r}, X_{j,r}). \tag{37}$$

Expressing $Cov(\overline{X}_{i,r}, \overline{X}_{j,r})$ as

$$Cov(\overline{X}_{i,r}, \overline{X}_{j,r}) = E\left[\overline{X}_{i,r}\overline{X}_{j,r}\right] - E\left[\overline{X}_{i,r}\right] E\left[\overline{X}_{j,r}\right]$$

we can simplify this equation in the following manner.

$$\begin{aligned}
Cov(\overline{X}_{i,r}, \overline{X}_{j,r}) &= E\left[(1-X_{i,r})(1-X_{j,r})\ \right] - E\left[(1-X_{i,r})\right] E\left[(1-X_{j,r})\right] \\
&= E\left[1 - X_{i,r} - X_{j,r} + X_{i,r}X_{j,r}\right] - (1 - E[X_{i,r}])\,(1 - E[X_{j,r}]) \\
&= 1 + E[X_{i,r}X_{j,r}] - E[X_{i,r}] - E[X_{j,r}] - \\
&\qquad\qquad (1 - E[X_{i,r}] - E[X_{j,r}] + E[X_{i,r}]E[X_{j,r}]) \\
&= Cov(X_{i,r}, X_{j,r}) \qquad (38)
\end{aligned}$$

This also implies that

$$Var(X_{i,r}) = Cov(X_{i,r}, X_{i,r}) = Cov(\overline{X}_{i,r}, \overline{X}_{i,r}) = Var(\overline{X}_{i,r}). \qquad (39)$$

Substituting equation (38) and (39) in equation (37) we get equation (36). □

Expressions for $Var(\overline{X}_{i,r})$ and $Cov(\overline{X}_{i,r}, \overline{X}_{j,r})$ are given in [5]. Substituting these in equation (36) we get

$$Var(C_{r,n}) = \sum_{i=1}^{N}(1-p_i)^r + 2\sum_{i}\sum_{<j}[(1-p_i-p_j)^r - (1-p_i)^r(1-p_j)^r].$$

When N is large, it is unlikely that $\tilde{p}$ for the inputs of a circuit will be available. In that case some approximation for this sampling process needs to be considered.

6 Problem 3: Distribution of the Number of Samples for a Fixed Pattern Coverage

In the previous section we derived the probability distribution for pattern coverage as a function of N the population size, r the number of samples and j the pattern coverage. In this section we address the problem of predicting the number of samples (or equivalently the number of test cycles) required to obtain a specific pattern coverage.

Let R be the random variable representing the number of samples required to obtain a pattern coverage of j (out of N). Then we can say that the probability of requiring exactly

r samples to achieve the specified coverage of j ($r \geq j$), denoted by $Q_{r,j}$, is given by

$$Q_{r,j} = \frac{N-(j-1)}{N} P_{0,j-1,r-1} \tag{40}$$

where $P_{0,j,r}$ is as given in equation (21). The RHS of Equation (40) is the probability that in $(r-1)$ clock cycles exactly $(j-1)$ distinct patterns were sampled and in the rth clock cycle a new pattern is sampled. Our aim is to obtain expressions for the average number of test cycles required to obtain the specified pattern coverage.

The probability generating function *pgf* of R, denoted by $h_j(y)$ is given by

$$h_j(y) = E\left[y^R\right] = \sum_{r=0}^{\infty} y^r \, P\{R=r\} = \sum_{r=0}^{\infty} y^r \, Q_{r,j}. \tag{41}$$

Substituting Equation (40) into Equation (41) we get

$$\begin{aligned} h_j(y) &= \sum_{r=0}^{\infty} y^r \frac{N-(j-1)}{N} P_{0,j-1,r-1} \\ &= \frac{(N-(j-1))y}{N} \sum_{r=0}^{\infty} y^{r-1} P_{0,j-1,r-1} \\ &= \frac{(N-(j-1))y}{N} g_{0,j-1}(y) \end{aligned} \tag{42}$$

where $g_{0,j-1}(y)$ is the generating function of $P_{0,j-1,r}$ with respect to r given by Equation (18). Substituting further from Equation (18) into Equation (42) we get

$$h_j(y) = \frac{y^j (N)_j}{N(N-y)(N-2y)\cdots(N-(j-1)y)}. \tag{43}$$

Since expected value of R (given N and j) is given by

$$E[R \mid N, j] = \left. \frac{d \, E[y^R]}{dy} \right|_{y=1}$$

we can use equation (43) to express the expected value of R as

$$E[R \mid N, j] = \frac{j! \binom{N}{j} y^{j-1}}{N(N-y)(N-2y)\cdots(N-(j-1)y)}$$

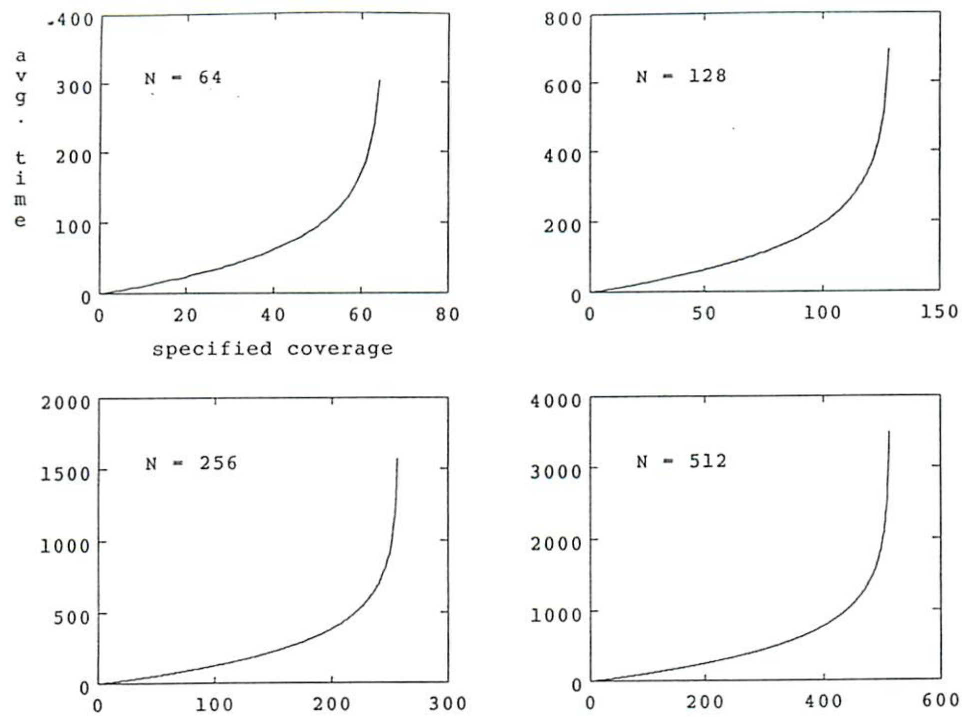

Figure 8: Expected number of clock cycles required to achieve specified pattern coverage.

$$\left(j + \frac{y}{N-y} + \frac{2y}{N-2y} + \cdots + \frac{(j-1)y}{N-(j-1)y} \right) \bigg|_{y=1}$$

$$= \quad j + \sum_{k=1}^{j-1} \frac{k/N}{(1-k/N)}. \tag{44}$$

Figure 8 shows the increase in the expected number of clock cycles required to obtain a specified pattern coverage j as j increases. It can be seen that the amount of effort required to obtain a specified coverage j increases in an approximately linear fashion for small values of j but becomes exponential for larger values.

7 Conclusion

In this paper we have presented an analysis for the efficiency of testing circuits using random patterns. As proposed by Kim et al. in [6] and Krasniewski and Pilarski in [8], we consider pattern coverage as a measure of test efficiency. This analysis showed that pattern sampling distributions can be partially ordered in terms of their effectiveness in capturing input patterns in a given number of clock cycles.

We have also provided a complete analysis for the distribution of pattern coverage for unbiased sampling of input patterns. In doing so it was shown that pattern coverage prediction can be viewed as the classical urn occupancy problem. Although the obtained expressions (for pattern coverage distribution) are computationally unwieldy, the practical utility of such an analysis arises from the fact that relations among various moments can be derived as shown in section 5.1.1. This makes it easy to get simple measures of test efficiency without extensive computations.

Further, using the above mentioned expressions, we have also obtained the distribution for the number of clock cycles required to obtain a specified coverage. The average value of this quantity is used as a measure of the difficulty of testing a given circuit.

For unequiprobable pattern sampling distributions, we have provided the expressions for the expected pattern coverage and the variance of pattern coverage as a function of the number of test cycles and the sampling distribution. Further work needs to be done to approximate such a sampling process and obtain expressions for its moments.

In order to obtain confidence bounds on test quality, we have presented two approximations for pattern coverage distribution. The *Laplace* approximation to Stirling's numbers of the second kind as stated in section 5.1.2 was derived in [2]. In this paper we have presented the derivation of a binomial approximation for the classical urn occupancy problem. Using these approximations confidence levels for achieving a specified pattern coverage can be easily obtained.

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