Factored Edge-Valued Binary Decision Diagrams And Their Application To Matrix Representation and Manipulation

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Abstract

Factored Edge-Valued Binary Decision Diagrams form an extension to Edge-Valued Binary Decision Diagrams. By associating both an additive and a multiplicative weight with the edges, FEVBDDs can be used to represent a wider range of functions concisely. As a result, the computational complexity for certain operations can be significantly reduced compared to EVBDDs. Additionally, the introduction of multiplicative edge weights allows us to directly represent the so-called complement edges which are used in OBDDs, thus providing a one to one mapping of all OBDDs to FEVBDDs. Applications such as integer linear programming and logic verification that have been proposed for EVBDDs also benefit from the extension. We present a complete matrix package based on FEVBDDs and apply the package to the problem of solving the Chapman-Kolmogorov equations.

Chapter 1

Introduction

Over the past decade a drastic increase in the integration of VLSI chips has taken place. Consequently, the complexity of the circuit designs has risen dramatically so that today's circuit designers rely more and more on sophisticated computer-aided design (CAD) tools. The ultimate goal of CAD tools is to automatically transform a description in the algorithmic or behavioral domains to the physical domain, i.e. down to the mask for chip production.

We divide this process into five different levels: *algorithmic, system, behavioral, logic* and *physical design*. So far CAD tools have been primarily used in the lower levels such as *physical design* and *logic* level.

At the logic level, the behaviour of the circuit is described by boolean functions. The efficiency of the algorithms applied in this level depends largely on the chosen data structure. Originally, representations such as the sum of products form or kernel representation were predominant. Today, the most popular data structure for boolean functions is the Ordered Binary Decision Diagram (OBDD) which provides a compact and canonical representation. In the wake of the successful introduction of the concept of function graphs by OBDDs, various other function graphs have been proposed. Function graphs such as Edge-Valued Binary Decision Diagrams and Multi Terminal Binary Decision Diagrams are not constrained to boolean functions but can be used to denote arithmetic functions. Since function graphs have proved extremely successful in the boolean logic level, efforts have been made to apply them to higher levels of abstraction. As an example, function graphs have been used for state reduction in finite state machines and logic verification of higher-level specifications. Additionally, function graphs have been applied to problems outside CAD, such as integer linear programming and matrix representation. The concept of function graphs may also be extended to model constructs of programming languages such as loop statements and recursive functions. The goal is to formally prove the equivalence of a sequential machine and the original specification.

Chapter 2

Function Graphs

Since the introduction of Ordered Binary Decision Diagrams (OBDD) by R. E. Bryant [6], several different forms of function graphs have been proposed. Functional Decision Diagrams (FDD) have been presented as an alternative to OBDDs for representing boolean functions [3]. Ordered Kronecker Functional Decision Diagrams (OKFDD) have been proposed by R. Drechsler, et al., [11] as a generalization of OBDDs and FDDs. Multi-Terminal Binary Decision Diagrams (MTBDD) [10] have been proposed to represent integer valued functions and extended to functions on finite sets [2]. Edge-Valued Binary Decision Diagrams (EVBDD) [14][15][16], and their extension to Factored Edge-Valued Binary Decision Diagrams (FEVBDD) provide a more compact means of representing such functions. Recently Binary Moment Diagrams (BMD and *BMD) [8] were introduced which permit efficient word-level verification of arithmetic functions.

2.1 Characteristics of Function Graphs

The various function graphs listed in Table 2.1 share some important common properties. All function graphs are represented as rooted, directed acyclic graphs (DAG) $(V \cup T, E)$ where V denotes the set of nonterminal nodes and T represents the set of terminal nodes. E stands for the set of edges. A strict variable ordering is imposed in order to guarantee a canonical

	Boolean	*Numeric or Symbolic*		
		Terminal	*Edge Weighted*	*Combined*
Pointwise	BDD	MTBDD	EVBDD, FEVBDD	
Moment	FDD	BMD		*BMD

Table 2.1: Categorization of Function Graphs

representation of the function. Table 2.1 supplies a categorization of the various function graphs according to the method of decomposition used and the range of the function.

There are two basically different methods to decompose a function with respect to a boolean variable x_i. The first method, called *pointwise decomposition*[1] , decomposes the function in terms of its positive and negative cofactors with respect to x_i (f_{x_i} and $f_{\overline{x}_i}$, respectively). The *positive cofactor* is the restriction of f to $x_i = 1$ whereas the *negative cofactor* corresponds to the restriction of f to $x_i = 0$. The second method, called *moment decomposition*, expresses the function in terms of its constant and linear moments ($f_{\overline{x}_i}$ and $f_{\dot{x}_i}$, respectively). The *constant moment* represents the value of the function with x set to 0 and is thus equivalent to the negative cofactor. The *linear moment* represents the change in f as a function of the change in x.

Additionally, we distinguish the function graphs by being either edge-weighted or having multiple terminal nodes. OBDDs and FDDs feature only two different terminal nodes 0 and 1, that denote the boolean values *true* and *false*, respectively. MTBDDs or ADDs represent the distinct function values by having a distinct terminal node for each such value. BMDs also encode the function values using multiple terminal nodes. Both EVBDDs and FEVBDDs associate weights with the edges in order to encode the function values. They have only one terminal node. *BMDs are hybrid function graphs in the sense that they have edge weights as well as multiple terminal nodes. The latter three function graphs require certain rules

[1]The terms *pointwise* and *moment decomposition* have been introduced in [8].

on how to compute the edge weights in order to preserve canonicity. Different rules lead to different edge weights when representing the same function. Thus, a function graph is uniquely defined by its basic decomposition rule and the rule for weight selection. In general, function representations based on an edge weighted encoding of the function values are more compact.

The increased compactness does come at a price, however. Problems that are solved in linear time in the graph size if OBDDs or MTBDDs are used may require traversing all possible paths in edge weighted function graphs. An example is that of deciding whether f evaluates to an arbitrary value a. Function graphs based on the moment decomposition rule suffer from yet another shortcoming. Evaluation of the function for a specific variable assignment generally requires traversing all nodes in the graph (cf. section 2.6). This is in contrast with evaluation of function graphs based on the pointwise decomposition rule which requires traversing only a single path for the given variable assignment. These problems are discussed in detail in the subsequent sections.

2.2 Function Decompositions

As mentioned above, function graph representations rely on a recursive decomposition of the function one variable at a time according to a given variable ordering. This is contrary to other function representations like the tabular form, ring sum of products form, sum of products form and monomial form where all the variables are expanded simultaneously.

How do these recursive function decompositions correspond to the representation of the function in function graphs? Since both the graph and the decomposition are defined recursively there is a natural relation between the two. Every node in the function graph denotes a function. Its two children or subnodes directly correspond to the two subfunctions obtained during the recursive decomposition process. That is the two children either denote the cofactors or the moments of the function with respect to the node variable.

Table 2.2 lists the various function decompositions. All these decomposition rules are

	Function Decomposition
BDD	$f = \overline{x} \wedge f_{\overline{x}} \vee x \wedge f_x$
FDD	$f = f_{\overline{x}} \oplus x \wedge (f_x \oplus f_{\overline{x}})$
MTBDD	$f = \overline{x} \wedge f_{\overline{x}} \vee x \wedge f_x$
ADD	
BMD	$f = f_{\overline{x}} + x \cdot (f_x - f_{\overline{x}}) = f_{\overline{x}} + x \cdot f_{\dot{x}}$
*BMD	$f = w \cdot (f_{\overline{x}} + x \cdot f_{\dot{x}})$
EVBDD	$f = x \cdot (f_x + v) + (1 - x) \cdot f_{\overline{x}}$
FEVBDD	$f = x \cdot (w \cdot f_x + v) + (1 - x) \cdot f_{\overline{x}}$

Table 2.2: Various Function Decompositions

derived from the well-known Shannon expansion [24] of a function f around a variable x_i:

$$f = x_i \wedge f|_{x_i=1} \vee \overline{x}_i \wedge f|_{x_i=0} \tag{2.1}$$

Obviously the OBDD expansion is a direct adaptation of equation (2.1). The FDD expansion is equal to the Reed-Muller expansion. It is obtained by applying the boolean identities

$$x \vee y = x \oplus y \oplus (x \wedge y) \tag{2.2}$$

$$\overline{x} = x \oplus 1 \tag{2.3}$$

to the Shannon expansion. Since the remaining expansions have non-boolean range, the Shannon expansion has to be modified in order to accommodate numeric values. This is achieved by replacing the boolean operations '$\wedge$' and '$\vee$' by their arithmetic counterparts '$\cdot$' and '+', respectively, and the negated decision variable $\overline{x_i}$ by $(1 - x_i)$. This represents a valid transformation since the decision variable x_i remains binary. Thus,

$$f = x_i \cdot f_{x_i} + (1 - x_i) \cdot f_{\overline{x}_i} \tag{2.4}$$

Here $+$, $-$ and $\cdot$ denote the basic arithmetic operations addition, subtraction and multiplication, respectively. In general, functions from the boolean domain $\{0, 1\}^n$ to a finite set S (such as the set of integers) are referred to as *pseudo-boolean* functions.

boolean	arithmetic
$\neg x$	$1-x$
$x \vee y$	$x+y-xy$
$x \wedge y$	xy
$x \oplus y$	$x+y-2xy$

Table 2.3: Arithmetic equivalents of boolean functions

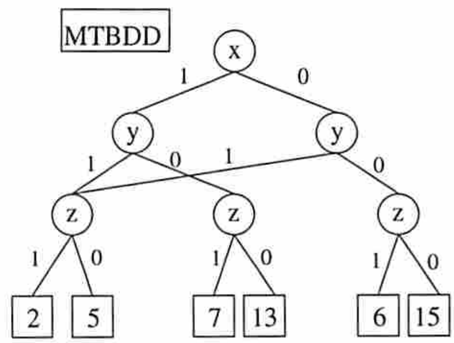
As shown in table 2.3, for these functions all boolean operations can be implemented using only arithmetic operations. Even though the value range of function f is now the integer domain, the decision variables still have to be binary variables $x_i \in \{0,1\}^n$. Thus, we can directly write a linear expression for a function f based on its function table. As an example, we have built the various function graphs (cf. figure 2.1) based on the different decompositions of the following function f.

$$
\begin{aligned}
f(x,y,z) &= 15\,(1-x)\,(1-y)\,(1-z) + 6\,(1-x)\,(1-y)\,z + \\
&\quad 5\,(1-x)\,y\,(1-z) + 2\,(1-x)\,y\,z + \\
&\quad 13\,x\,(1-y)\,(1-z) + 7\,x\,(1-y)\,z + \\
&\quad 5\,x\,y\,(1-z) + 2\,x\,y\,z & (2.5)\\
&= 15 - 2x - 9z - 10y + 2xy + 3xz + 6yz - 3xyz & (2.6)\\
&= (3-2y)(5-3z) + x(y-1)(2-3z) & (2.7)\\
&= 15 + x(-2 + y(-8 + z(-3)) + (1-y)(z(-6))) + & (2.8)\\
&\quad (1-x)(y(-10 + z(-3)) + (1-y)(z(-9)))\\
&= 15 - 9(x(\frac{2}{9} + \frac{2}{3}(y(\frac{4}{3} + \frac{1}{2}z) + (1-y)z)) + & (2.9)\\
&\quad (1-x)(y(\frac{10}{9} + \frac{1}{3}z) + (1-y)z))
\end{aligned}
$$

Equation (2.5) is in a form that directly corresponds to the function decomposition for MTB-DDs or ADDs and the tabular form. The monomial form of f (cf. equation (2.6)) is derived from equation (2.5) by expanding the expression and collecting common terms. The coefficients of the monomial form of f are reflected in the values of the terminal nodes in the BMD representation of f. Equation (2.7) shows the extraction of common linear terms in the monomial form and corresponds to the *BMD. Equations (2.8) and (2.9) reflect the structure of the decomposition rules for EVBDDs and FEVBDDs, respectively. The different function graphs are shown in figure 2.1. The characteristics of these function decompositions are explained in more detail in the subsequent sections.

Tabular Form

x	y	z	f
0	0	0	15
0	0	1	6
0	1	0	5
0	1	1	2
1	0	0	13
1	0	1	7
1	1	0	5
1	1	1	2



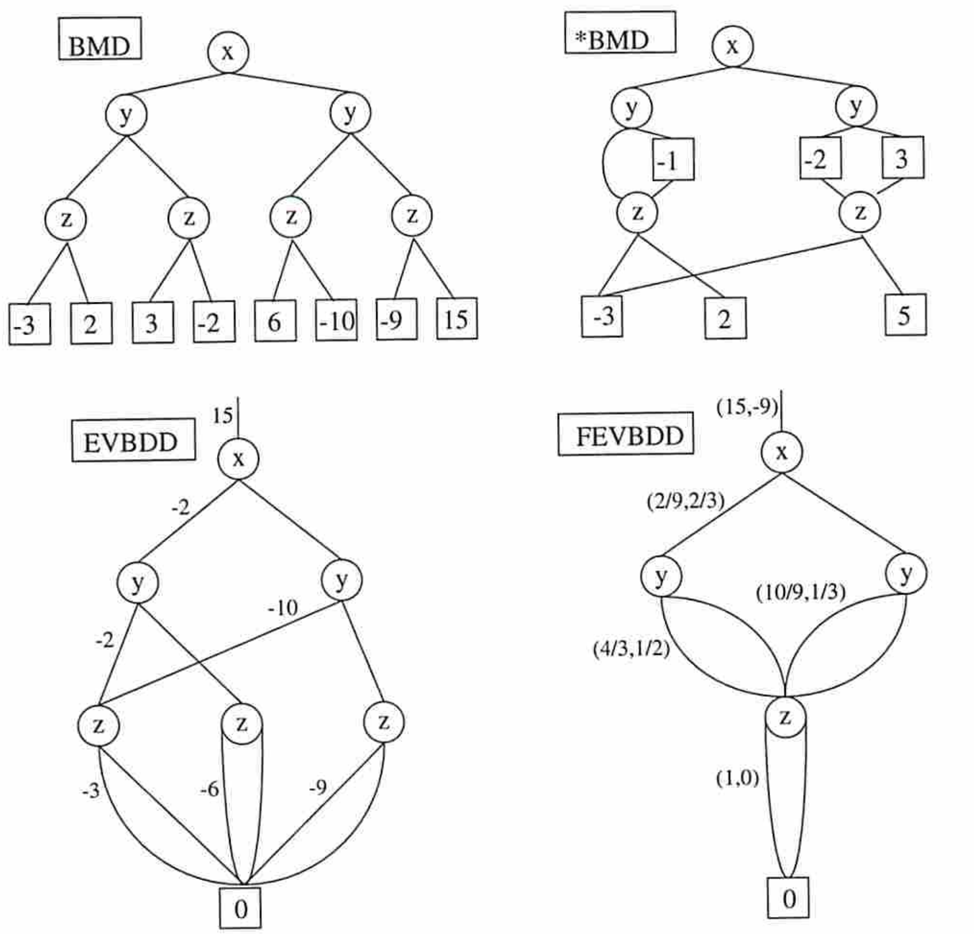


Figure 2.1: Example for function decompositions and function graphs

2.3 Ordered Binary Decision Diagrams OBDD

Ordered Binary Decision Diagrams were proposed in 1986 by Bryant [6]. By imposing a strict ordering on the decision variables of the Binary Decision Diagram introduced by Lee [18] and further popularized by Akers [1], they offer a compact and canonical symbolic representation of boolean functions. Since then OBDDs have been widely used for various applications such as logic verification, probabilistic analysis, finite-state system analysis and many more.

Definition 2.1 *[6] An OBDD is a rooted, directed acyclic graph* $(V \cup T, E)$ *consisting of two types of vertices.*

- *A nonterminal vertex* $\mathbf{v} \in V$ *is represented by a triple* $\langle variable(\mathbf{v}), child_t(\mathbf{v}), child_e(\mathbf{v})\rangle$ *where* $variable(\mathbf{v}) \in \{x_0, \ldots, x_{n-1}\}$.
- *A terminal vertex* $\mathbf{v} \in T$ *has as an attribute a* $value(\mathbf{v}) \in \{0, 1\}$.

There is no nonterminal vertex $\mathbf{v}$ *such that* $child_t(\mathbf{v}) = child_e(\mathbf{v})$, *and there are no two nonterminal vertices* $\mathbf{u}$ *and* $\mathbf{v}$ *such that* $\mathbf{u} = \mathbf{v}$. *Furthermore, there exists an index function* $index(x) \in \{0, \ldots, n-1\}$ *such that the following holds for every nonterminal vertex. If* $child_t(\mathbf{v})$ *is also nonterminal, then we must have* $index(variable(\mathbf{v})) < index(variable(child_t(\mathbf{v})))$. *If* $child_e(\mathbf{v})$ *is nonterminal, then we must have* $index(variable(\mathbf{v})) < index(variable(child_e(\mathbf{v})))$. *The function* f *denoted by* $\langle x, \mathbf{v}_t, \mathbf{v}_e\rangle$ *is* $f = x \wedge f_t \vee \overline{x} \wedge f_e$ *where* f_t *and* f_e *are the functions denoted by* $\mathbf{v}_t$ *and* $\mathbf{v}_e$, *respectively. The functions denoted by* 0 *and* 1 *are the constant function* $\mathbf{0}$ *and* $\mathbf{1}$, *respectively.*

OBDDs are evaluated according to the following definition.

Definition 2.2 *Given an OBDD node* $\mathbf{v}$ *representing* $f(x_0, \ldots, x_{n-1})$ *and a function* Φ *that for each variable* x *assigns a value* $\Phi(x)$ *equal to either 0 or 1, the function OBDDeval is* *defined as*

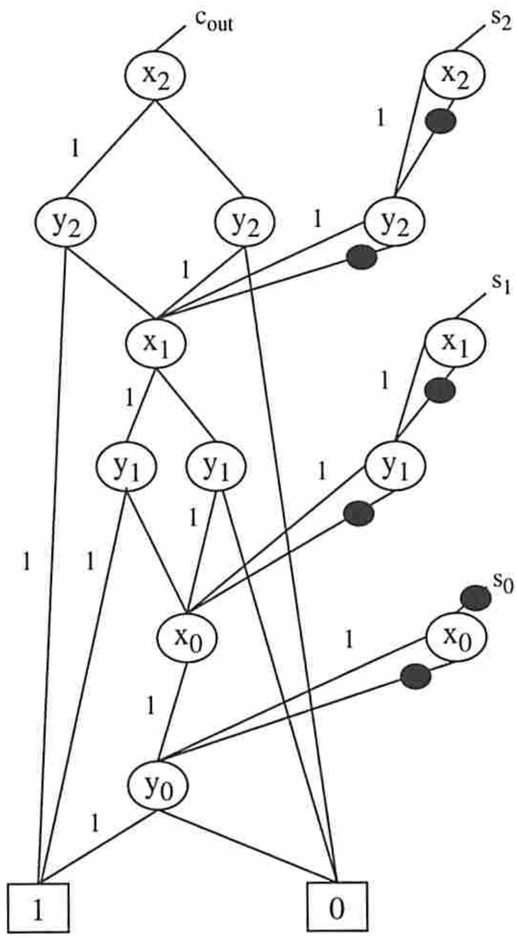
$$OBDDeval(\mathbf{v}, \Phi) = \begin{cases} value(\mathbf{v}) & \mathbf{v} \textit{ is a terminal node} \\ OBDDeval(child_t(\mathbf{v}), \Phi) & \Phi(variable(\mathbf{v})) = 1 \\ OBDDeval(child_e(\mathbf{v}), \Phi) & \Phi(variable(\mathbf{v})) = 0 \end{cases}$$

To compute all common binary boolean operations an algorithm called *apply* has been devised by Bryant [6] and it has been shown that $|f \circ g| \leq |f| \cdot |g|$ holds, i.e. the time complexity of operations on OBDDs is $O(|f| \cdot |g|)$.

The size of OBDDs is potentially exponential but it has been established that for many applications the OBDD size is manageable. In general, the size of OBDDs is strongly dependent on the variable ordering. For example, the two functions $f_1 = x_1 \wedge x_2 \vee \ldots \vee x_{2n-1} \wedge x_{2n}$ and $f_2 = x_1 \wedge x_{n+1} \vee \ldots \vee x_n \wedge x_{2n}$ only differ in their variable indexing but whereas f_1 is represented by an OBDD of size $2n+2$, f_2 requires a graph of size 2^{n+1} if we use $x_1 < x_2 < \ldots < x_{2n}$ as the variable ordering [6]. Therefore, various heuristics to determine good variable orderings have been suggested [12][19]. Mostly these heuristics are static. Another approach to finding a good variable ordering is the so-called 'sifting'-algorithm for dynamic variable ordering proposed by Rudell [23].

Another means to improve the memory efficiency of an OBDD package is the concept of complement edges [4] which allows both a function and its complement to be represented by the same graph. A complement edge is an ordinary edge with an extra bit (complement bit) set to indicate that the connected formula is to be interpreted as the complement of the ordinary formula. An example of an OBDD containing complement edges is given in figure 2.2. The OBDD represents a 3-bit adder. If the complement function is selected the branch leading to the subgraph has to be complemented. Since there exist functional equivalences for different assignments of complement edges to a node a set of rules [4] has to be obeyed in order to maintain canonicity. OBDDs are mainly denoted using the ternary boolean operator ITE (short for if–then–else) [4][6].

$$ITE(F, G, H) = F \wedge G \vee \overline{F} \wedge H \tag{2.10}$$



OBDD with complement edges

Figure 2.2: OBDD representation of the four output functions of a 3-bit adder

The ITE operation can be used to implement all two-variable boolean operations. Since the ITE operation translates directly to the logical functions performed in each vertex, it is often better to express boolean operations on the OBDD in terms of the ITE operation. For example, the Shannon decomposition of $Z = ITE(F, G, H)$ expressed in ITE operations is as follows.

$$\begin{aligned} Z &= v \wedge Z_v \vee \overline{v} \wedge Z_{\overline{v}} \\ &= ITE(v, ITE(F_v, G_v, H_v), ITE(F_{\overline{v}}, G_{\overline{v}}, H_{\overline{v}})) \end{aligned} \tag{2.11}$$

2.4 Multi Terminal Binary Decision Diagrams MTBDD

In an effort to represent functions with a non-boolean value range OBDDs have been extended to Multi Terminal Binary Decision Diagrams (MTBDD)[2] which feature more than two different terminal nodes [2][9][10]. Throughout the following sections of this paper we will exclusively use the term MTBDD to cover both MTBDDs and ADDs.

Definition 2.3 *[10] [2] An MTBDD is a rooted, directed acyclic graph* $(V \cup T, E)$, *representing a function* $f : \{0,1\}^n \rightarrow S$ *where* S *is a finite set.* V *is the set of nonterminal vertices* $\mathbf{v}$. T *is the set of terminal vertices.*

- *A nonterminal vertex* $\mathbf{v} \in V$ *is represented by a triple* $\langle variable(\mathbf{v}), child_t(\mathbf{v}), child_e(\mathbf{v})\rangle$ *where* $variable(\mathbf{v}) \in \{x_0, \ldots, x_{n-1}\}$.
- *A terminal vertex* $\mathbf{v} \in T$ *has as an attribute a* $value(\mathbf{v}) \in S$.

There is no nonterminal node $\mathbf{v}$ *such that* $child_t(\mathbf{v}) = child_e(\mathbf{v})$, *and there are no two nonterminal vertices* $\mathbf{u}$ *and* $\mathbf{v}$ *such that* $\mathbf{u} = \mathbf{v}$. *Furthermore, there exists an index function* $index(x) \in \{0, \ldots, n-1\}$ *such that the following holds for every nonterminal vertex. If* $child_t(\mathbf{v})$ *is also nonterminal, then we must have* $index(variable(\mathbf{v})) < index(variable(child_t(\mathbf{v})))$. *If* $child_e(\mathbf{v})$

[2]MTBDDs are also referred to as Algebraic Decision Diagrams ADD [2]. They are the same as the flattened form of EVBDDs [16].

is nonterminal, then we must have $index(variable(\mathbf{v})) < index(variable(child_e(\mathbf{v})))$. *The function* f *denoted by* $\langle x, \mathbf{v}_t, \mathbf{v}_e \rangle$ *is* $f = x \wedge f_t \vee \overline{x} \wedge f_e$ *where* f_t *and* f_e *are the functions denoted by* $\mathbf{v}_t$ *and* $\mathbf{v}_e$, *respectively.*

MTBDDs can also be represented by an array of OBDDs. In the case of an MTBDD representing integer valued functions the function f is then expressed as a summation over boolean functions f_i:

$$f = \sum_{i=1}^{n} f_i \cdot 2^i \tag{2.12}$$

where each f_i has value 0 or 1 and is represented by an OBDD [9].

Definition 2.4 *Given an MTBDD node* $\mathbf{v}$ *representing* $f(x_0, \ldots, x_{n-1})$ *and a function* Φ *that for each variable* x *assigns a value* $\Phi(x)$ *equal to either 0 or 1, the function MTBDDeval is defined as*

$$MTBDDeval(\mathbf{v}, \Phi) = \begin{cases} value(\mathbf{v}) & \mathbf{v} \textit{ is a terminal node} \\ MTBDDeval(child_t(\mathbf{v}), \Phi) & \Phi(variable(\mathbf{v})) = 1 \\ MTBDDeval(child_e(\mathbf{v}), \Phi) & \Phi(variable(\mathbf{v})) = 0 \end{cases}$$

MTBDDs form a canonical representation of functions from the boolean domain $\{0,1\}^n$ to a finite set S. An example for an MTBDD representing such a pseudo-boolean function has been given in figure 2.1. Since complementation in the sense of boolean algebra may not have a meaningful counterpart in S the concept of complement edges is mostly not applicable. So far MTBDDs have proven to be useful in such applications as spectral transformations of boolean functions [9]. They have also been applied to such fields as efficient representation of matrices [10][2], shortest path algorithm etc. [2].

2.5 Edge-Valued Binary Decision Diagrams EVBDD

Edge-Valued Binary Decision Diagrams,which were proposed by Lai, et al., [14][15][16] offer a direct extension to the concept of OBDDs. By associating a so-called edge value *ev* to every then-edge of the OBDD they are capable of representing pseudo-boolean functions such as integer valued functions. Their application has proven successful in such areas as formal verification [14] and integer linear programming [17], spectral transformation, and function decomposition [15].

Definition 2.5 *[16] An EVBDD is a tuple* $\langle c, \mathbf{f} \rangle$ *where* c *is a constant value and* $\mathbf{f}$ *is a rooted, directed acyclic graph* $(V \cup T, E)$ *consisting of two types of vertices.*

- *A nonterminal vertex* $\mathbf{f} \in V$ *is represented by a quadruple* $\langle variable(\mathbf{f}), child_t(\mathbf{f}), child_e(\mathbf{f}), ev \rangle$*, where* $variable(\mathbf{f}) \in \{x_0, \ldots, x_{n-1}\}$ *is a binary variable.*

- *The single terminal vertex* $\mathbf{f} \in T$ *with value 0 is denoted by* $\mathbf{0}$.

There is no nonterminal vertex $\mathbf{f}$ *such that* $child_t(\mathbf{f}) = child_e(\mathbf{f})$ *and* $ev = 0$*, and there are no two nonterminal vertices* $\mathbf{f}$ *and* $\mathbf{g}$ *such that* $\mathbf{f} = \mathbf{g}$*. Furthermore, there exists an index function* $index(x) \in \{0, \ldots, n-1\}$ *such that the following holds for every nonterminal vertex. If* $child_t(\mathbf{f})$ *is also nonterminal, then we must have* $index(variable(\mathbf{f})) < index(variable(child_t(\mathbf{f})))$*. If* $child_e(\mathbf{f})$ *is nonterminal, then we must have* $index(variable(\mathbf{f})) < index(variable(child_e(\mathbf{f})))$*.*

Definition 2.6 *[16][16] An EVBDD* $\langle c, \mathbf{f} \rangle$ *denotes the arithmetic function* $c + f : \{0,1\}^n \rightarrow$ *integer where* f *is the function* f *denoted by* $\mathbf{f} = \langle x, \mathbf{f}_t, \mathbf{f}_e, ev \rangle$*. The terminal node* $\mathbf{0}$ *represents the constant function* $f = 0$*, and* $\langle x, \mathbf{f}_t, \mathbf{f}_e, ev \rangle$ *denotes the arithmetic function* $f = x \cdot (ev + f_t) + (1 - x) \cdot f_e$*.*

Definitions (2.5), (2.6) provide a graphical representation of pseudo-boolean functions. As a consequence integer variables have to be encoded in binary as in $X = \sum_{i=0}^{n-1} x_i \cdot 2^i$ where X is a n-bit integer variable. It has been shown that EVBDDs form a canonical representation of arithmetic functions.

Definition 2.7 *Given an EVBDD* $\langle c, \mathbf{f} \rangle$ *representing* $f(x_0, \ldots, x_{n-1})$ *and a function* Φ *that for each variable* x *assigns a value* $\Phi(x)$ *equal to either 0 or 1, the function EVBDDeval is defined as*

$$EVBDDeval(\langle c, \mathbf{f} \rangle, \Phi) = \begin{cases} c & \mathbf{f} \textit{ is the terminal node } \mathbf{0} \\ EVBDDeval(\langle c + ev, child_t(\mathbf{f}) \rangle, \Phi) & \Phi(variable(\mathbf{f})) = 1 \\ EVBDDeval(\langle c, child_e(\mathbf{f}) \rangle, \Phi) & \Phi(variable(\mathbf{f})) = 0 \end{cases}$$

An example for a pseudo-boolean function represented by an EVBDD is given in figure 2.1.

Boolean functions can be represented in EVBDDs by using the integers 0 and 1 to denote the boolean values *true* and *false*. Boolean operations are implemented through arithmetic operations as shown in table 2.3. A method has been proposed by Lai, et al., that converts any OBDD representation of a boolean function to its corresponding EVBDD representation [16][16]. It can be proven that both function graphs OBDD $\mathbf{v}$ and EVBDD $\langle c, \mathbf{v}' \rangle$ denoting the same function f share the same topology except that the terminal node **1** is absent from the EVBDD and the edges connected to it are redirected to the single terminal node **0**. Additionally, it was shown that boolean operations executed on EVBDDs have the same time complexity $O(|f| \cdot |g|)$ as boolean operations on OBDDs. The concept of complement edges can not be realized in EVBDDs.

As has been done for OBDDs, a generic operation *apply* can be defined that implements arbitrary arithmetic operations on the EVBDD representations $\langle c_f, \mathbf{f} \rangle$, and $\langle c_g, \mathbf{g} \rangle$ of two arithmetic functions f and g. In general, the time complexity of such an operation on two EVBDDs $\langle c_f, \mathbf{f} \rangle$, and $\langle c_g, \mathbf{g} \rangle$ is $O(\|\langle c_f, \mathbf{f} \rangle\| \cdot \|\langle c_g, \mathbf{g} \rangle\|) = O(|\langle c_f, \mathbf{f}' \rangle| \cdot |\langle c_g, \mathbf{g}' \rangle|$ where $\langle c_f, \mathbf{f}' \rangle$, and $\langle c_g, \mathbf{g}' \rangle$ denote the flattened EVBDDs of $\langle c_f, \mathbf{f} \rangle$ $\langle c_g, \mathbf{g} \rangle$, respectively. A flattened EVBDD is defined in exactly the same manner as an MTBDD. For operations such as addition, subtraction, scalar-multiplication, etc. the time complexity of *apply* can be drastically reduced by exploiting certain properties. A scalar multiplication $c \cdot (c_f + f)$ can be done with time complexity

$O(|\langle c_f, \mathbf{f}\rangle|)$ by simply multiplying all edge values by c. All operations op, such as addition, that fulfill the *additive property*

$$\langle c_f, \mathbf{f}\rangle op \langle c_g, \mathbf{g}\rangle = \langle c_f op c_g, \mathbf{f}\, op \mathbf{g}\rangle \tag{2.13}$$

have the reduced time complexity $O(|\langle c_f, \mathbf{f}\rangle| \cdot |\langle c_g, \mathbf{g}\rangle|)$.

Based on EVBDDs a concept of structured EVBDDs (SEVBDDs) has been developed [16]. SEVBDDs allow the modeling of conditional expressions and vectors. Their main application lies in the field of formal verification.

2.6 Binary Moment Diagrams: BMD and *BMD

Binary Moment Diagrams BMD and their extension to Multiplicative Binary Moment Diagrams *BMD were proposed recently by Bryant and Chen [8]. They are based on an adaptation of the Reed-Muller-decomposition (cf. table 2.2) of boolean functions to the integer domain. The decomposition rule for BMDs is easily obtained by rearranging the terms in the integer version of the Shannon expansion (cf. equation 2.4):

$$\begin{aligned} f &= x \cdot f_x + (1-x) \cdot f_{\overline{x}} \\ &= f_{\overline{x}} + x \cdot (f_x - f_{\overline{x}}) \\ &= f_{\overline{x}} + x \cdot f_{\dot{x}} \end{aligned} \tag{2.14}$$

The term $f_{\overline{x}}$ is referred to as the *constant moment* of f with respect to x since it denotes the portion of function f that is independent of x. The term $f_{\dot{x}}$ is called its *linear moment* with respect to x.

Definition 2.8 *[8] A BMD is a rooted, directed acyclic graph $(V \cup T, E)$, representing a function $f : \mathcal{Z}^n \rightarrow S$ where $\mathcal{Z}$ is any set of integers. V is the set of nonterminal vertices $\mathbf{v}$. T is the set of terminal vertices.*

- *A nonterminal vertex $\mathbf{v} \in V$ is represented by a triple $\langle variable(\mathbf{v}), child_l(\mathbf{v}), child_c(\mathbf{v})\rangle$ where $variable(\mathbf{v}) \in \{x_0, \ldots, x_{n-1}\}$ is a numeric variable.*

- *A terminal vertex $v \in T$ has as an attribute a $value(\mathbf{v}) \in S$.*

There is no nonterminal vertex $\mathbf{v}$ such that $child_l(\mathbf{v}) = 0$. There are no two nonterminal vertices $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u} = \mathbf{v}$. Furthermore, there exists an index function $index(x) \in \{0, \ldots, n-1\}$ such that the following holds for every nonterminal vertex. If $child_l(\mathbf{v})$ is also nonterminal, then we must have $index(variable(\mathbf{v})) < index(variable(child_l(\mathbf{v})))$. If $child_c(\mathbf{v})$

is nonterminal, then we must have $index(variable(\mathbf{v})) < index(variable(child_c(\mathbf{v})))$*. The function* f *denoted by* $\langle x, \mathbf{v}_l, \mathbf{v}_c \rangle$ *is* $f = f_c + x \cdot f_l$ *where* f_c *and* f_l *are the functions denoted by* $\mathbf{v}_c$ *and* $\mathbf{v}_l$*, respectively.*

Definition 2.9 *Given an BMD node* $\mathbf{v}$ *representing* $f(x_0, \ldots, x_{n-1})$ *and a function* Φ *that for each variable* x *assigns a value* $\Phi(x) \in \mathcal{Z}$*, the function BMDeval is defined as*

$$BMDeval(\mathbf{v}, \Phi) =$$
$$= \begin{cases} value(\mathbf{v}) & \mathbf{v} \text{ is a terminal node} \\ BMDeval(child_c(\mathbf{v}), \Phi) + & \text{otherwise} \\ \quad \Phi(variable(\mathbf{v})) \cdot BMDeval(child_l(\mathbf{v}), \Phi) & \end{cases}$$

Definition 2.10 *[8] A *BMD is a tuple* $\langle w, \mathbf{v}, rule \rangle$ *where* w *is a constant value,* $\mathbf{v}$ *is a rooted, directed acyclic graph* $(V \cup T, E)$*, consisting of two types of vertices, and* rule *is the set of weight normalizing rules applied to the graph. The *BMD represents a function* $f : \mathcal{Z}^n \to S$ *with* $\mathcal{Z}$ *being any set of numbers.*

- *A nonterminal vertex* $\mathbf{v} \in V$ *is represented by a 5-tuple* $\langle variable(\mathbf{v}), child_l(\mathbf{f}), child_c(\mathbf{f}), w_l, w_c \rangle$*, where* $variable(\mathbf{f}) \in \{x_0, \ldots, x_{n-1}\}$ *is a numeric variable.*
- *A terminal vertex* $\mathbf{v} \in T$ *has as an attribute a* $value(\mathbf{v}) \in S$*.*

There is no nonterminal vertex $\mathbf{v}$ *such that* $child_l(\mathbf{v}) = 0$ *. There are no two nonterminal vertices* $\mathbf{u}$ *and* $\mathbf{v}$ *such that* $\mathbf{f} = \mathbf{g}$*. Furthermore, there exists an index function* $index(x) \in \{0, \ldots, n-1\}$ *such that the following holds for every nonterminal vertex. If* $child_l(\mathbf{v})$ *is also nonterminal, then we must have* $index(variable(\mathbf{v})) < index(variable(child_l(\mathbf{v})))$*. If* $child_c(\mathbf{v})$ *is nonterminal, then we must have* $index(variable(\mathbf{v})) < index(variable(child_c(\mathbf{v})))$*.*

*BMDs form an extension to the BMD concept using the idea of edge weights introduced for EVBDDs by Lai, et al. [16]. By introducing multiplicative edge weights they allow the

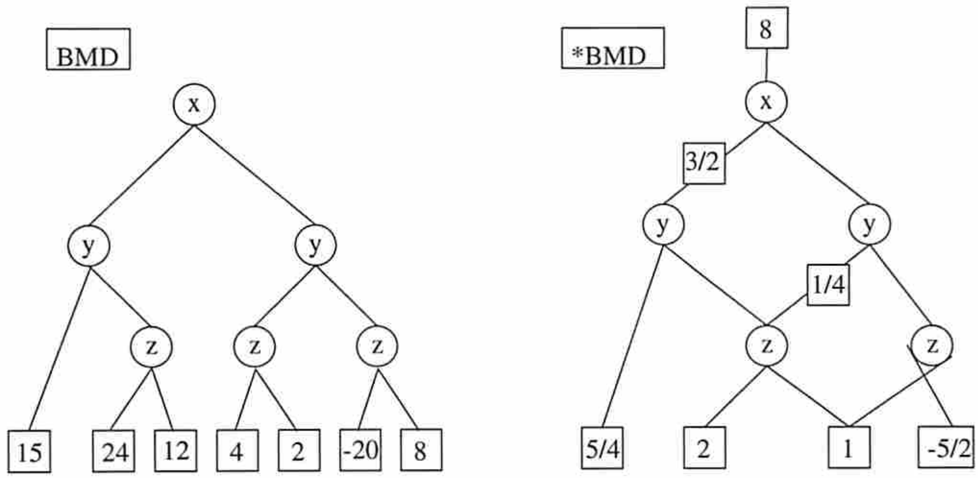


Figure 2.3: BMD and *BMD for $f = 8 - 20z + 2y + 4yz + 12x + 24xz + 15xy$.

sharing of subgraphs that are multiples of a common subgraph. Figure 2.3 shows both the BMD and the *BMD for the linear function $f = 8 - 20z + 2y + 4yz + 12x + 24xz + 15xy$.

Definition 2.11 *Given a *BMD* $\langle w, \mathbf{v}, rule\rangle$ *representing* $f(x_0, \ldots, x_{n-1})$ *and a function* Φ *that for each variable* x *assigns a value* $\Phi(x) \in \mathcal{Z}$, *the function *BMDeval is defined as*

$$*BMDeval(\langle w, \mathbf{f}, rule\rangle, \Phi) =$$
$$= \begin{cases} value(\mathbf{v}) & \mathbf{v} \textit{ is a terminal node} \\ w \cdot (\, *BMDeval(\langle w_c, child_c(\mathbf{v}), rule\rangle, \Phi) + & \textit{otherwise} \\ \quad \Phi(variable(\mathbf{v})) \cdot *BMDeval(\langle w_l, child_l(\mathbf{v}), rule\rangle, \Phi)) & \end{cases}$$

As stated in definition (2.10) *BMDs require a set of rules for normalizing the edge weights in order to guarantee a strong canonical form of the function graph. These rules are applied when generating a new node given nodes $\mathbf{v_l}$ and $\mathbf{v_c}$. Based on the chosen rule the weight w of the top node is computed and the weights w_l and w_c are readjusted. Consequently two graphs representing the same function but using two different rules share the same basic topology but differ in the values of their edge weights. So far two rules have been proposed [8]. Given two *BMDs $\langle w_l, \mathbf{v_l}, rule_l\rangle$ and $\langle w_c, \mathbf{v_c}, rule_c\rangle$ with $rule_l = rule_c = rule$ the node weight w of $\langle w, \mathbf{v}, rule\rangle$ is computed as follows:

Factored Edge-Valued Binary Decision Diagrams And Their Application To Matrix Representation and Manipulation

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1. integer rule:

$$\begin{array}{lcl} w & = & \gcd(w_l, w_c) \\ sign(w) & = & sign(w_c) \end{array}$$

2. rational rule:

$$w = \begin{cases} w_c & \text{if } w_c \neq 0 \\ w_l & \text{otherwise} \end{cases}$$

As can be seen in the definitions of *BMDeval* and **BMDeval*, respectively, BMDs as well as *BMDs are not constrained to representing pseudo-boolean functions but can be regarded as representing a restricted class of linear functions, that is, those functions f that can be expressed as a sum of monomial terms, or those that satisfy equation (2.14) for all variables. Addition of linear functions can easily be defined as $h = f + g$ such that $h(\Phi) = f(\Phi) + g(\Phi)$ for all variable assignments Φ. The set of linear functions $\mathcal{L}$ and the addition operation form a group $\langle \mathcal{L}, + \rangle$. Since the product '$\cdot$' of two linear functions might yield a quadratic function a *linear product* '$\hat{\cdot}$' is introduced so that $\mathcal{L}$ is closed under '$\hat{\cdot}$'.

$$f \hat{\cdot} g = f_{\overline{x}} \hat{\cdot} g_{\overline{x}} + x(f_{\overline{x}} \hat{\cdot} g_{\dot{x}} + f_{\dot{x}} \hat{\cdot} g_{\overline{x}} + f_{\dot{x}} \hat{\cdot} g_{\dot{x}}) \tag{2.15}$$

The algebraic structure $\langle \mathcal{L}, +, \hat{\cdot} \rangle$ forms a ring [8]. Under restrictions such as the boolean domain restriction ($\Phi(x) \in \{0, 1\}$) or the independent support assumption (f and g have disjoint support sets) it holds that $f \cdot g = f \hat{\cdot} g$. Under the boolean domain restriction it further holds that any operation can be linearized such that $[f \hat{op} g](\Phi) = f(\Phi) op g(\Phi)$ [8].

Even though *BMDs are capable of associating a numeric variable with every node, integers are usually encoded in binary for applications such as logic verification. This is due to the fact that such applications need to represent functions in both the boolean and integer domain. All commonly used encodings of signed integers such as sign-magnitude, two's complement, and one's complement can be easily expressed using *BMDs. Figure 2.4 shows the *BMD representations of these encodings. This property in combination with a very compact representation of word level operations (cf. table 3.5) make *BMDs a powerful means for symbolic

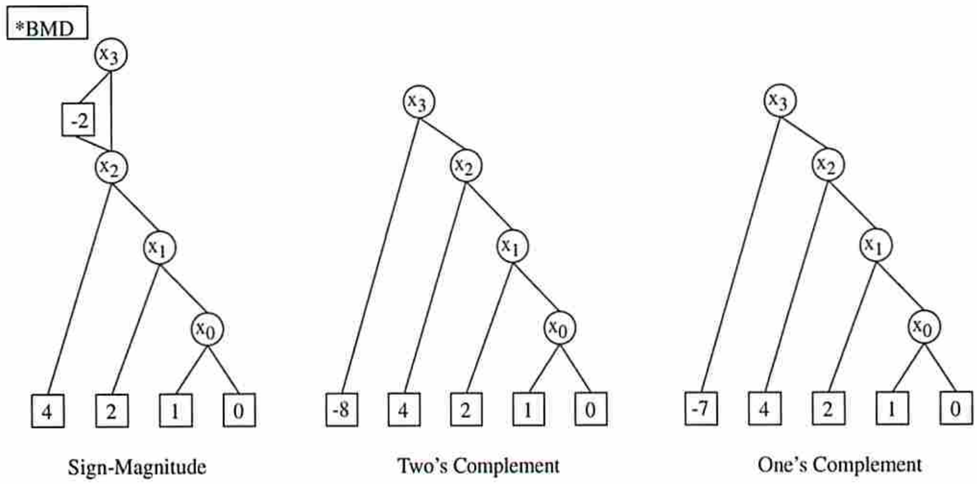

Figure 2.4: Representations of signed integers using *BMDs

representation of functions in the word level domain. Representations of word-level sum and product using *BMDs are presented in figures 3.3 and 3.4 in section 3.3.

Of course there is a trade off between the improved space complexity in terms of the graph size and the increased time complexity of evaluating the graph. As can be seen easily from the definitions of *BMDeval* and **BMDeval*, a naive implementation of the evaluation rule is potentially exponential. This is due to the fact that for every assignment $\Phi(x) = 1$ both successors of the node have to be evaluated. Thus, for the worst case that $\Phi(x) = 1, \forall x$, all possible paths in the graph have to be traversed. On the contrary, function graphs obeying a pointwise decomposition rule require the traversal of exactly one path when evaluating the function for any given assignment Φ. One can improve the performance of the evaluation rule by making use of an *evaluation cache*. This evaluation cache stores the function value for an already evaluated subgraph. Contrary to the caches used for storing results of operations on function graphs, this cache can only be used during the current evaluation process. It has to be flushed after the evaluation is finished. If we employ an evaluation cache during evaluation of a function, the complexity is reduced to $O(|f|)$, for any given assignment. A model algorithm for evaluating *BMDs using an evaluation cache is presented in table 2.4. This is still worse

```
*BMDeval+(⟨w_v, v, rule_v⟩, Φ)
{
    if(v ∈ T)
        return(value(v));
    *
    * has the current node already been evaluated ?
    *
    if(eval_table_lookup(⟨w_v, v, rule_v⟩, Φ(variable(v)), ans))
        return(ans);
    ans = w · ( *BMDeval+(⟨w_c, child_c(v), rule⟩, Φ)+
        Φ(variable(v)) · *BMDeval+(⟨w_l, child_l(v), rule⟩, Φ));
    *
    * store the result in the cache
    *
    eval_table_insert(⟨w_v, v, rule_v⟩, Φ(variable(v)), ans)
    return(ans);
}
```

Table 2.4: **BMDeval+*

than the complexity of an evaluation on MTBDD or EVBDD, which is $O(supp(f))$.

No formal relation between the representation of boolean functions in OBDDs and *BMDs could be established, as it has been done for EVBDDs and FEVBDDs. Although all boolean functions that have been dealt with in applications of *BMDs so far are reportedly of similar size as the corresponding OBDD, it is expected that there are cases where this is not valid. This is a consequence of the fact that OBDDs (as well as EVBDDs and FEVBDDs) are based on a pointwise decomposition whereas *BMDs rely on a moment decomposition. In the case of FDDs, which are also decomposed momentwise, an example has been shown where the FDD size is exponential while the OBDD size is polynomial in the number of supported variables [3].

In contrast to the single generic *apply* algorithm established for OBDDs and EVBDDs, *BMDs require specifically tailored apply algorithms for adding and multiplying two *BMDs

f and g. The worst case complexity for the *PlusApply* algorithm on *BMDs is reported to be $O(m_f \cdot m_g)$ where m_f and m_g denote the sizes of the corresponding BMDs of f and g, respectively. For *MultApply* the complexity issue is even more severe since every call to *MultApply* requires four recursive calls to *MultApply* as well as two calls to *PlusApply*.

The complexity of boolean operations on *BMDs is also a major drawback. Despite the similar graph sizes for *BMD and OBDD, the complexity of boolean operations on the *BMD is potentially much higher. As has been done for OBDDs a generic apply algorithm for boolean operations *BinaryApply* has been proposed [8]. It is based on a dynamic reconstruction of the positive and negative cofactors f_x, and $f_{\bar{x}}$ from the linear and constant moments f_x, and $f_{\dot{x}}$ as follows:

$$[f\hat{op}g]_{\bar{x}} = f_{\bar{x}}\hat{op}g_{\bar{x}} \tag{2.16}$$

$$\begin{aligned}[f\hat{op}g]_{\dot{x}} &= [f_x\hat{op}g_x] - [f_{\bar{x}}\hat{op}g_{\bar{x}}] \\ &= [(f_{\bar{x}} + f_{\dot{x}})\hat{op}(g_{\bar{x}} + g_{\dot{x}})] - [f_{\bar{x}}\hat{op}g_{\bar{x}}]\end{aligned} \tag{2.17}$$

It is obvious that every recursion step of the generic binary apply based on equation (2.17) requires two calls to *BinaryApply* and three calls to *PlusApply*. To improve the computational complexity it is better to exploit properties of the operation that needs to be executed. For boolean '*and*', an implementation based on equation (2.15), i.e. *MultApply*, can be given that offers improved computational complexity. Still the complexity of boolean operations executed on BMDs is potentially drastically increased compared to its OBDD based counterpart. This is particularly true for operations such as boolean '*or*' and '*xor*'. If we want to compute these operations on *BMDs we have to use either *BinaryApply* or algorithms based on the following definitions.

$$\overline{f} = 1 - f \tag{2.18}$$

$$f \wedge g = f\hat{\cdot}g \tag{2.19}$$

$$f \vee g = f + g - f\hat{\cdot}g \tag{2.20}$$

	$\neg$	$\wedge$	$\vee$	$\oplus$
*BMD	$O(\lvert m_f \rvert)$	$\gg O(\lvert f\rvert \cdot \lvert g\rvert)$	$\gg O(\lvert f\rvert \cdot \lvert g\rvert)$	$\gg O(\lvert f\rvert \cdot \lvert g\rvert)$
FEVBDD	const.	$O(\lvert f\rvert \cdot \lvert g\rvert)$	$O(\lvert f\rvert \cdot \lvert g\rvert)$	$O(\lvert f\rvert \cdot \lvert g\rvert)$

Table 2.5: Computational complexity of boolean operations on *BMDs and FEVBDDs

$$f \oplus g \;=\; f + g - 2f\hat{\cdot}g \tag{2.21}$$

As can be seen in equation (2.20) performing boolean '*or*' requires two calls to *PlusApply* and another call to *MultApply*. Equation (2.21) shows that the same applies for boolean '*xor*'. Thus, boolean '*or*' and boolean '*xor*' behave worse than boolean '*and*' in terms of their computational complexity. Table 2.5 shows a comparison between the computational complexity of boolean operations on *BMDs and FEVBDDs. In this table m_f denotes the size of the corresponding BMD. As has been mentioned before, in many cases the graphs of the *BMD representation and the OBDD representation are of comparable size. As is explained in section 3.2, FEVBDDs allow a mapping of an OBDD containing complement edges to an FEVBDD with the same number of internal nodes. The concept of complement edges cannot be realized directly in *BMDs, however. Therefore, we can expect that the size of the FEVBDD is even somewhat smaller than the size of the corresponding *BMD.

In summary,BMDs and *BMDs in particular offer a powerful representation of arithmetic functions. They outperform other function graphs in terms of the graph size for all word-level operations except for addition. In the boolean domain, however, they face the challenge of a high complexity for boolean operations. Their range of application is therefore primarily located in the word-level domain. They have been successfully applied to the problem of verifying multipliers of up to 62-bit word size [8].

Chapter 3

Factored Edge-Valued Binary Decision Diagrams

Factored Edge-Valued Binary Decision Diagrams (FEVBDD) are an extension to EVBDDs introduced by Lai, et al., [14][15][16]. EVBDDs have been discussed in section 2.5. By associating both an additive and a multiplicative weight with the true-edges[1] FEVBDDs offer a more compact representation of linear functions, since common subfunctions differing only by an affine transformation can now be expressed by a single subgraph. Additionally, they allow the concept of complement edges to be transferred from OBDDs to FEVBDDs.

Definition 3.1 *An FEVBDD is a tuple* $\langle c, w, \mathbf{f}, rule\rangle$ *where* c *and* w *are constant values,* $\mathbf{f}$ *is a rooted, directed acyclic graph* $(V \cup T, E)$ *consisting of two types of vertices, and rule is the set of weight normalizing rules applied to the graph.*

- *A nonterminal vertex* $\mathbf{f} \in V$ *is represented by a 6-tuple*[2] $\langle variable(\mathbf{f}), child_t(\mathbf{f}), child_e(\mathbf{f}), ev, w_t, w_e\rangle$*, where* $variable(\mathbf{f}) \in \{x_0, \ldots, x_{n-1}\}$ *is a binary variable.*

[1]The GCD rule requires also a multiplicative weight to be associated with the else-edges.

[2]If we use the rational rule it holds that $w_e = 1$ for all nodes. Thus we can represent a nonterminal vertex by a 5-tuple $\langle variable(\mathbf{f}), child_t(\mathbf{f}), child_e(\mathbf{f}), ev, w_t\rangle$.

- *The single terminal vertex* $\mathbf{f} \in T$ *with value 0 is denoted by* $\mathbf{0}$. *By definition all branches leading to* $\mathbf{0}$ *have an associated weight* $w = 0$.

There is no nonterminal vertex $\mathbf{f}$ *such that* $child_t(\mathbf{f}) = child_e(\mathbf{f})$, $ev = 0$, *and* $w_e = w_t = 1$, *and there are no two nonterminal vertices* $\mathbf{f}$ *and* $\mathbf{g}$ *such that* $\mathbf{f} = \mathbf{g}$. *Furthermore, there exists an index function* $index(x) \in \{0, \ldots, n-1\}$ *such that the following holds for every nonterminal vertex. If* $child_t(\mathbf{f})$ *is also nonterminal, then we must have* $index(variable(\mathbf{f})) < index(variable(child_t(\mathbf{f})))$. *If* $child_e(\mathbf{f})$ *is nonterminal, then we must have* $index(variable(\mathbf{f})) < index(variable(child_e(\mathbf{f})))$.

Definition 3.2 *A FEVBDD* $\langle c_f, w_f, \mathbf{f}, rule_f \rangle$ *denotes the arithmetic function* $c_f + w_f \cdot f$ *where* f *is the function* f *denoted by* $\mathbf{f} = \langle x, \mathbf{f}_t, \mathbf{f}_e, ev, w_t, w_e \rangle$. *The terminal node* $\mathbf{0}$ *represents the constant function* $f = 0$, *and* $\langle x, \mathbf{f}_t, \mathbf{f}_e, ev, w_t, w_e \rangle$ *denotes the arithmetic function* $f = x \cdot (ev + w_t \cdot f_t) + (1 - x) \cdot w_e \cdot f_e$.

Definition 3.3 *Given a FEVBDD* $\langle c_f, w_f, \mathbf{f}, rule_f \rangle$ *representing* $f(x_0, \ldots, x_{n-1})$ *and a function* Φ *that for each variable* x *assigns a value* $\Phi(x)$ *equal to either 0 or 1, the function FEVBDDeval is defined as*

$$FEVBDDeval(\langle c_f, w_f, \mathbf{f}, rule_f \rangle, \Phi) =$$
$$= \begin{cases} c_f & \mathbf{f} \text{ is the terminal node } \mathbf{0} \\ c_f + w_f \cdot FEVBDDeval(\langle ev, w_t, child_t(\mathbf{f}), rule \rangle, \Phi) & \Phi(variable(\mathbf{f})) = 1 \\ c_f + w_f \cdot FEVBDDeval(\langle 0, w_e, child_e(\mathbf{f}), rule \rangle, \Phi) & \Phi(variable(\mathbf{f})) = 0 \end{cases}$$

An example for a FEVBDD denoting a pseudo-boolean function is given in figure 2.1.

Lemma 3.1 *Given two FEVBDDs* $\langle c_f, w_f, \mathbf{f}, rule_f \rangle$ *and* $\langle c_g, w_g, \mathbf{g}, rule_g \rangle$, *which have been generated using the same weight normalizing rule and with* f *and* g *being non-isomorphic, it holds that there exists an assignment* $\Phi \in \{0,1\}^n$ *such that* $c_f + w_f \cdot f \neq c_g + w_g \cdot g$ *for this assignment.*

Proof:

case1: if $c_f \neq c_g$ then let $\Phi = \mathbf{0}$; it follows that $FEVBDDeval(\langle c_f, w_f, \mathbf{f}, rule_f\rangle, \Phi) = c_f \neq FEVBDDeval(\langle c_g, w_g, \mathbf{g}, rule_g\rangle, \Phi) = c_g$.

case2: $c_f = c_g$ and $w_f = w_g$; by the definition of non-isomorphism it holds that $\exists\Phi$ such that $FEVBDDeval(\langle 0, 1, \mathbf{f}, rule\rangle, \Phi) \neq FEVBDDeval(\langle 0, 1, \mathbf{g}, rule\rangle, \Phi)$. Consequently, we have that $c_f + w_f \cdot f \neq c_g + w_g \cdot g$ for this assignment Φ.

case3: $c_f = c_g$ and $w_f \neq w_g$; we assume that it holds that f and g are non-isomorphic and that $c_f + w_f \cdot f = c_g + w_g \cdot g$ for all assignments Φ. This implies that $w_f \cdot f = w_g \cdot g$ or $f = \frac{w_g}{w_f} \cdot g$. Consequently, f and g are isomorphic which contradicts the original assumption. Thus, it holds that $\exists\Phi$ such that $c_f + w_f \cdot f \neq c_g + w_g \cdot g$. □

Theorem 3.1 *Two FEVBDDs $\langle c_f, w_f, \mathbf{f}, rule_f\rangle$ and $\langle c_g, w_g, \mathbf{g}, rule_g\rangle$ that have been generated using the same weight normalizing rule, i.e. $rule_f = rule_g$, denote the same function, i.e. $\forall\Phi \in \{0,1\}^n, FEVBDDeval(\langle c_f, w_f, \mathbf{f}, rule_f\rangle, \Phi) = FEVBDDeval(\langle c_g, w_g, \mathbf{g}, rule_g\rangle, \Phi)$, if and only if $c_f = c_g$, $w_f = w_g$, and* **f** *and* **g** *are isomorphic.*

Proof:

Sufficiency: If $c_f = c_g$ and $w_f = w_g$ and **f** and **g** are isomorphic, then $\forall\Phi$, $FEVBDDeval(\langle c_f, w_f, \mathbf{f}, rule_f\rangle, \Phi) = FEVBDDeval(\langle c_g, w_g, \mathbf{g}, rule_g\rangle, \Phi)$ follows directly from the definitions of isomorphism and *FEVBDDeval*.

Necessity: If $c_f \neq c_g$ then let $\Phi = \mathbf{0}$; it holds that $FEVBDDeval(\langle c_f, w_f, \mathbf{f}, rule_f\rangle, \Phi) = c_f \neq c_g = FEVBDDeval(\langle c_g, w_g, \mathbf{g}, rule_g\rangle, \Phi)$. If $c_f = c_g$ and $w_f \neq w_g$ then let Φ be an arbitrary assignment such that $FEVBDDeval(\langle 0, 1, \mathbf{f}, rule\rangle, \Phi) \neq 0$ and $FEVBDDeval(\langle 0, 1, \mathbf{g}, rule\rangle, \Phi) \neq 0$; it holds that $FEVBDDeval(\langle c_f, w_f, \mathbf{f}, rule_f\rangle, \Phi) = c_f + w_f \cdot FEVBDDeval(\langle 0, 1, \mathbf{f}, rule\rangle, \Phi)$ and $FEVBDDeval(\langle c_g, w_g, \mathbf{g}, rule_g\rangle, \Phi) = c_g + w_g \cdot FEVBDDeval(\langle 0, 1, \mathbf{g}, rule\rangle, \Phi)$. If f and g are isomorphic then it holds by the definition of isomorphism and *FEVBDDeval* that $FEVBDDeval(\langle 0, 1, \mathbf{f}, rule\rangle, \Phi) = FEVBDDeval(\langle 0, 1, \mathbf{g}, rule\rangle, \Phi) = val$. It follows that $c_f +$

$w_f \cdot val \neq c_g + w_g \cdot val$. If f and g are non-isomorphic lemma 3.1 holds. Now we have to prove the lemma for the last condition **f** being isomorphic to **g**. We need to show that if **f** and **g** are not isomorphic, then $\exists \Phi \in \{0,1\}^n$ such that $\mathit{FEVBDDeval}(\langle 0, 1, \mathbf{f}, rule\rangle, \Phi) \neq \mathit{FEVBDDeval}(\langle 0, 1, \mathbf{g}, rule\rangle, \Phi)$. Without loss of generality, we assume $\mathit{index}(\mathit{variable}(\mathbf{f})) \leq \mathit{index}(\mathit{variable}(\mathbf{g}))$. Let $k = n - \mathit{index}(\mathit{variable}(\mathbf{f}))$, we will prove the lemma by induction on k.

Base: If $k = 0$, both **f** and **g** are terminal nodes. Furthermore, $\mathbf{f} = \mathbf{g} = \mathbf{0}$. Thus, **f** and **g** are isomorphic.

Induction hypothesis: Assume the above holds for $n - \mathit{index}(\mathit{variable}(\mathbf{f})) < k$.

Induction: We show that the hypothesis holds for $n - \mathit{index}(\mathit{variable}(\mathbf{f})) = k$. Let $\mathbf{f} = \langle x_{n-k}, \mathbf{f_t}, \mathbf{f_e}, ev_f, w_{t_f}, w_{e_f}\rangle$.

case 1: $n - \mathit{index}(\mathit{variable}(\mathbf{g})) = k$, i.e. $\mathbf{g} = \langle x_{n-k}, \mathbf{g_t}, \mathbf{g_e}, ev_g, w_{t_g}, w_{e_g}\rangle$.

If $ev_f \neq ev_g$ then let $\Phi(x_{n-k}) = 1$ and $\Phi(x_i) = 0, \forall i \neq n-k$. Then it holds that $\mathit{FEVBDDeval}(\langle 0, 1, \mathbf{f}, rule\rangle, \Phi) = ev_f \neq ev_g = \mathit{FEVBDDeval}(\langle 0, 1, \mathbf{g}, rule\rangle, \Phi)$. If $ev_f = ev_g$ and $w_{t_f} \neq w_{t_g}$ then let Φ be an arbitrary assignment such that $\Phi(x_{n-k}) = 1$ and $\mathit{FEVBDDeval}(\langle 0, 1, \mathbf{f}, rule\rangle, \Phi) \neq 0$ and $\mathit{FEVBDDeval}(\langle 0, 1, \mathbf{f}, rule\rangle, \Phi) \neq 0$. Then it holds that $\mathit{FEVBDDeval}(\langle 0, 1, \mathbf{f}, rule\rangle, \Phi) = ev_f + w_{t_f} \cdot \mathit{FEVBDDeval}(\langle 0, 1, \mathbf{f_t}, rule\rangle, \Phi)$ and $\mathit{FEVBDDeval}(\langle 0, 1, \mathbf{g}, rule\rangle, \Phi) = ev_g + w_{t_g} \cdot \mathit{FEVBDDeval}(\langle 0, 1, \mathbf{g_t}, rule\rangle, \Phi)$. If $\mathbf{f_t}$ and $\mathbf{g_t}$ are isomorphic it holds that $\mathit{FEVBDDeval}(\langle 0, 1, \mathbf{f}, rule\rangle, \Phi) = \mathit{FEVBDDeval}(\langle 0, 1, \mathbf{g}, rule\rangle, \Phi) = val$. Thus we have that $ev_f + w_{t_f} \cdot val \neq ev_g + w_{t_g} \cdot val$. If $\mathbf{f_t}$ and $\mathbf{g_t}$ are nonisomorphic then lemma 3.1 is applicable. Almost the identical prove can be given for $ev_f = ev_g$ and $w_{e_f} \neq w_{e_g}$. If $ev_f = ev_g$, $w_{t_f} = w_{t_g}$, and $w_{e_f} = w_{e_g}$ either $\mathbf{f_t}$ and $\mathbf{g_t}$, or $\mathbf{f_e}$ and $\mathbf{g_e}$ are nonisomorphic.

subcase 1: If $\mathbf{f_t}$ and $\mathbf{g_t}$ are nonisomorphic, then from $n - \mathit{index}(\mathit{variable}(\mathbf{f_t})) < k$,$n - \mathit{index}(\mathit{variable}(\mathbf{g_t})) < k$, and the induction hypothesis, we see that there exists some

Φ such that $FEVBDDeval(\langle 0,1,\mathbf{f_t},rule\rangle,\Phi) \neq FEVBDDeval(\langle 0,1,\mathbf{g_t},rule\rangle,\Phi)$. Now let Φ' be defined as $\Phi'(x_{n-k}) = 1$ and $\Phi'(x_i) = \Phi(x_i), \forall i \neq n-k$, then $FEVBDDeval(\langle 0,1,\mathbf{f},rule\rangle,\Phi') = ev_f + w_{t_f} \cdot FEVBDDeval(\langle 0,1,\mathbf{f_t},rule\rangle,\Phi') \neq ev_g + w_{t_g} \cdot FEVBDDeval(\langle 0,1,\mathbf{g_t},rule\rangle,\Phi') = FEVBDDeval(\langle 0,1,\mathbf{g},rule\rangle,\Phi')$.

subcase 2: Otherwise $\mathbf{f_e}$ and $\mathbf{g_e}$ are nonisomorphic, then by similar arguments, letting $\Phi'(x_{n-k}) = 0$ and $\Phi'(x_i) = \Phi(x_i), \forall i \neq n-k$, will result in $FEVBDDeval(\langle 0,1,\mathbf{f},rule\rangle,\Phi') \neq FEVBDDeval(\langle 0,1,\mathbf{g},rule\rangle,\Phi')$.

case 2 : $n - index(variable(\mathbf{g})) < k$

By definition of a reduced FEVBDD, we cannot have $ev_f = 0$, $w_{t_f} = w_{e_f} = 1$ and $\mathbf{f_t}$ being isomorphic to $\mathbf{f_e}$. If $ev_f \neq 0$, let $\Phi(x_{n-k}) = 1$ and $\Phi(x_i) = 0$, $\forall i \neq n-k$, then $FEVBDDeval(\langle 0,1,\mathbf{f},rule\rangle,\Phi) = ev_f \neq 0 = FEVBDDeval(\langle 0,1,\mathbf{g},rule\rangle,\Phi)$, since $\mathbf{g}$ is independent of the first $n-k$ bits. If $ev_f = 0$ and $w_{t_f} \neq w_{e_f}$ then let Φ be an assignment such that $\Phi(x_{n-k}) = 1$ and $\Phi(x_i) = 0$, $\forall((i \neq n-k) \wedge (i < n - index(variable(\mathbf{g}))))$. Furthermore, let Φ be such that $FEVBDDeval(\langle 0,1,\mathbf{f},rule\rangle,\Phi) = val_f \neq 0$ and $FEVBDDeval(\langle 0,1,\mathbf{g},rule\rangle,\Phi) = val_g \neq 0$. If the corresponding subgraph of $\mathbf{f}$ with top-variable x_{n-k} and $\mathbf{g}$ are isomorphic then it holds that $val_f = w_{t_f} \cdot val_g$. If the graphs are non-isomorphic we can apply the same reasoning as we did in the proof of lemma 3.1. Otherwise, $\mathbf{f_t}$ and $\mathbf{f_e}$ are non-isomorphic and at least one of them is not isomorphic to $\mathbf{g}$. If $\mathbf{f_t}$ and $\mathbf{g}$ are non-isomorphic, then by induction hypothesis, there exists an assignment Φ such that $FEVBDDeval(\langle 0,1,\mathbf{f_t},rule\rangle,\Phi) \neq FEVBDDeval(\langle 0,1,\mathbf{g},rule\rangle,\Phi)$. Now, let $\Phi'(x_{n-k}) = 1$ and $\Phi'(x_i) = \Phi(x_i), \forall i \neq n-k$. It holds that $FEVBDDeval(\langle 0,1,\mathbf{f_t},rule\rangle,\Phi') \neq FEVBDDeval(\langle 0,1,\mathbf{g},rule\rangle,\Phi')$.□

As shown above, FEVBDDs form a canonical representation of a function only for specific weight normalizing rules that uniquely determine how the node weight of a new node is computed based on its both descendants. We propose two basic rules that can be used to guarantee canonicity for FEVBDDs. Given two FEVBDDs $\langle c_t, w_t, \mathbf{t}, rule_t \rangle$ and $\langle c_e, w_e, \mathbf{e}, rule_e \rangle$ with $rule_t = rule_e = rule$ the node weight w of $\langle c, w, \mathbf{f}, rule \rangle$ is computed as follows

1. GCD rule:

$$\begin{aligned} w &= \gcd(c_t - c_e, w_t, w_e) \\ &= \gcd(c_t - c_e, \gcd(w_t, w_e)) \\ sign(w) &= \begin{cases} sign(w_e) & \text{if } w_e \neq 0 \\ sign(w_t) & \text{if } w_t \neq 0 \wedge w_e = 0 \\ sign(c_t - c_e) & \text{if } w_t = w_e = 0 \end{cases} \end{aligned}$$

2. RATIONAL rule:

$$w = \begin{cases} w_e & \text{if } w_e \neq 0 \\ w_t & \text{if } w_t \neq 0 \wedge w_e = 0 \\ (c_t - c_e) & \text{if } w_t = w_e = 0 \end{cases}$$

These weight normalizing rules (cf. table 3.2) are applied whenever a new node is generated using the *make_new_node* routine (cf. table 3.1). This routine enforces both the canonicity of the function graph as well as its uniqueness. The routine *find_or_add* preserves the uniqueness of all nodes. Before a new node is actually created a quick hash table lookup is performed and, if the node is already a member of the table, the stored node with its unique ID is returned. Otherwise, a new node entry in the hash table is created and the new node with its unique ID is returned. Thus it is guaranteed that every node is stored only once in the hash table.

Though the GCD rule requires a multiplicative weight to be associated with both the true- and the else-edge, there are some cases where it might be the rule of choice. If the function range is purely integer the GCD rule avoids dealing with fractions. This is particularly valuable, since all arithmetic operations on fractions are significantly more time consuming than the built in hardware routines for integers. Furthermore, the restriction to integers by use of the GCD rule brings a clear advantage in memory efficiency (cf. table 4.3). Even though we need to store an additional weight, the memory consumption per node is less than when using the rational rule which requires the use of fractions. This is because every fraction is internally represented as one integer for the numerator and one for the denominator. Of course, as soon as the

```
make_new_node(x_i,⟨c_T, w_T, T, rule_T⟩,⟨c_E, w_E, E, rule_E⟩)
{
    if{⟨c_T, w_T, T, rule_T⟩ = ⟨c_E, w_E, E, rule_E⟩}
        return(⟨c_T, w_T, T, rule_T⟩);
    *
    * compute the new weights
    *
    c = c_E;
    ev = c_T − c_E;
    w = norm_weight(ev, w_T, w_E);
    w_t = w_T/w;
    w_e = w_E/w;
    *
    * guarantee uniqueness
    *
    ⟨c_h, w_h, h, rule_h⟩=find_or_add(x_i, c, w, ev, w_t, w_e, T, E);
    return ⟨c_h, w_h, h, rule_h⟩ ;
}
```

Table 3.1: *Make_New_Node*

```
norm_weight(ev, w_T, w_E)
{
    switch(mode) {
    case 'GCD':
        if(w_E ≠ 0)
            sign = sign(w_E);
        else if(w_T ≠ 0)
            sign = sign(w_T);
        else sign = sign(ev);
        return(sign · gcd(ev, w_T, w_E)) ;
    break ;
    case 'RATIONAL':
        if(w_E ≠ 0)
            return w_E;
        else if(w_T ≠ 0)
            return w_T;
        else return ev;
    break;
}
```

Table 3.2: *Norm_Weight*

application requires the use of fractions the rational rule should be preferred. Nevertheless, the GCD rule is still applicable since we define:

$$\gcd(\frac{u}{u'}, \frac{v}{v'}) = \frac{\gcd(u, v)}{\gcd(u', v')}$$

3.1 Operations

As has been done for OBDDs [6] and EVBDDs [16] we provide a generic algorithm *apply* that implements arbitrary arithmetic operations on two FEVBDDs (cf. table 3.3). Of course this algorithm does not offer the most efficient realization of some operations. We will later give improved versions for these operations. *Apply* takes two FEVBDDs $\langle c_f, w_f, \mathbf{f}, rule_f\rangle$ and $\langle c_g, w_g, \mathbf{g}, rule_g\rangle$, as well as an operation *op* as its arguments. Both FEVBDDs have to be based on the same weight normalizing rule. The algorithm recursively branches at the top variable, i.e. the variable with the least index in **f** or **g** until it reaches a terminal case. Terminal cases depend on the operation *op*; as an example, for *op*='+' we have the terminal case $\langle c_f, w_f, \mathbf{f}, rule_f\rangle + \langle c, 0, \mathbf{0}, rule\rangle$. The computational efficiency of this algorithm can be improved significantly by taking advantage of a computation cache. Before the recursive process is started, a quick lookup in the computation cache is performed and if successful, then the result of *op* is returned immediately without further computation. The entries of the cache are uniquely identified by a key consisting of the operands $\langle c_f, w_f, \mathbf{f}, rule_f\rangle$ and $\langle c_g, w_g, \mathbf{g}, rule_g\rangle$ and the operation *op*. Whenever a new result is computed it is stored in the computation cache. In general the complexity of operations performed by *apply* is $O(\|\langle c_f, w_f, \mathbf{f}, rule_f\rangle\| \cdot \|\langle c_g, w_g, \mathbf{g}, rule_g\rangle\|)$.

As mentioned before we can further improve the computational complexity of *apply* by making use of properties of specific operations. We adapt the concept of an *additive property* proposed for EVBDDs by Lai, et al., [16] and extend it to the so-called *affine property* for FEVBDDs.

```
apply(⟨c_f, w_f, f, rule_f⟩,⟨c_g, w_g, g, rule_g⟩,op) {
    *
    * check for a terminal case
    *
    if(terminal_case(⟨c_f, w_f, f, rule_f⟩,⟨c_g, w_g, g, rule_g⟩,op))
        return (⟨c_f, w_f, f, rule_f⟩op⟨c_g, w_g, g, rule_g⟩);
    *
    * is the result of op already available in the
    * computation cache
    *
    if(comp_table_lookup(⟨c_f, w_f, f, rule_f⟩,⟨c_g, w_g, g, rule_g⟩,op,⟨c_ans, w_ans, ans, rule_ans⟩))
        return (⟨c_ans, w_ans, ans, rule_ans⟩);
    *
    * perform the recursive computation of op
    *
    if(index(f) ≥ index(g)) {
        ⟨c_gt, w_gt, g_t, rule_gt⟩ = ⟨c_g + w_g · ev_g, w_g · w_tg, child_t(g), rule⟩;
        ⟨c_ge, w_ge, g_e, rule_ge⟩ = ⟨c_g, w_g · w_eg, child_e(g), rule⟩;
        var = variable(g);
    }
    else {
        ⟨c_gt, w_gt, g_t, rule_gt⟩ = ⟨c_ge, w_ge, g_e, rule_ge⟩ = ⟨c_g, w_g, g, rule_g⟩;
        var = variable(f);
    }
    if(index(f) ≤ index(g)) {
        ⟨c_ft, w_ft, f_t, rule_ft⟩ = ⟨c_f + w_f · ev_f, w_f · w_tf, child_t(f), rule⟩;
        ⟨c_fe, w_fe, f_e, rule_fe⟩ = ⟨c_f, w_f · w_ef, child_e(f), rule⟩;
    }
    else {
        ⟨c_ft, w_ft, f_t, rule_ft⟩ = ⟨c_fe, w_fe, f_e, rule_fe⟩ = ⟨c_f, w_f, f, rule_f⟩;
    }
    ⟨c_ht, w_ht, h_t, rule_ht⟩ = apply(⟨c_ft, w_ft, f_t, rule_ft⟩, ⟨c_gt, w_gt, g_t, rule_gt⟩, op);
    ⟨c_he, w_he, h_e, rule_he⟩ = apply(⟨c_fe, w_fe, f_e, rule_fe⟩, ⟨c_ge, w_ge, g_e, rule_ge⟩, op);
    if(⟨c_ht, w_ht, h_t, rule_ht⟩ = ⟨c_he, w_he, h_e, rule_he⟩ )
        return(⟨c_ht, w_ht, h_t, rule_ht⟩);
    ⟨c_h, w_h, h, rule_h⟩ = make_new_node(var, ⟨c_ht, w_ht, h_t, rule_ht⟩, ⟨c_he, w_he, h_e, rule_he⟩);
    *
    * store the result in the computation cache
    *
    comp_table_insert(⟨c_f, w_f, f, rule_f⟩,⟨c_g, w_g, g, rule_g⟩,op,⟨c_h, w_h, h, rule_h⟩);
    return (⟨c_h, w_h, h, rule_h⟩);
}
```

Table 3.3: *Apply*

Definition 3.4 *An operator op applied to* $\langle c_f, w_f, \mathbf{f}, rule_f \rangle$ *and* $\langle c_g, w_g, \mathbf{g}, rule_g \rangle$ *is said to satisfy the affine property if*

$$(c_f + w_f \cdot f) op (c_g + w_g \cdot g) = (c_f \ op \ c_g) + w \cdot ((w'_f \cdot f) op (w'_g \cdot g))$$

The factor w *is defined as* $w = \gcd(w_f, w_g)$ *and can be of arbitrary value.*[3]

Operations that satisfy the affine property are addition, subtraction, scalar multiplication and logical bit shifting. The main advantage of the affine property lies in reducing the computational complexity of *apply*. Since we can separately compute the parts of the result generated by the constants c_f and c_g and by the two subgraphs $\langle 0, w_f, \mathbf{f}, rule \rangle$ and $\langle 0, w_g, \mathbf{g}, rule \rangle$, the hit ratio of the computation cache can be drastically increased by separating the influence of the constants and always storing only the results for $c = 0$. This concept is applied to every recursion step so that the constant value is never passed down to the next recursion level. Unfortunately, we still have to pass the multiplicative weights w_f and w_g since they cannot be separated from the functions f and g. To achieve a further improvement in the hit ratio, we extract the common divisor w from w_f and w_g and promote only w'_f and w'_g. This is an advantage in such cases as reducing the problem of performing $(7 + 8 \cdot f) op (4 + 6 \cdot g)$ to the already computed problem $(0 + 4 \cdot f) op (0 + 3 \cdot g)$. Since we cannot quantify the influence of the GCD extraction the worst case computational complexity for operations satisfying the affine property is given as $O(|\langle c_{f'}, \mathbf{f}' \rangle| \cdot |\langle c_{g'}, \mathbf{g}' \rangle|)$ where $\langle c_{f'}, \mathbf{f}' \rangle$ and $\langle c_{g'}, \mathbf{g}' \rangle$ denote the EVBDDs corresponding to the FEVBDDs $\langle c_f, w_f, \mathbf{f}, rule_f \rangle$ and $\langle c_g, w_g, \mathbf{g}, rule_g \rangle$, respectively.

Scalar multiplication and logical-bit shifting offer a better computational complexity since they can be computed in time independent of the size of the function graph. Scalar multiplication only requires the weights of the root node to be multiplied. In the case of EVBDDs we

[3] Similar to the rational rule we can alternatively define the *affine property* as follows:

$$(c_f + w_f \cdot f) op (c_g + w_g \cdot g) = (c_f \ op \ c_g) + w_f \cdot (f op (\frac{w_g}{w_f} \cdot g)).$$

All the benefits of the *affine property* remain the same.

have to multiply every edge weight with the scalar; a task of complexity $O(|\mathbf{f}|)$.

Since multiplication does not satisfy the affine property we are basically required to use the original version of *apply*. For the multiplication of two functions that both have a high percentage of reconverging branches the following approach might improve the cache efficiency.

$$\begin{aligned}\langle c_f, w_f, \mathbf{f}, rule_f\rangle \cdot \langle c_g, w_g, \mathbf{g}, rule_g\rangle &= \langle c_f \cdot c_g, 0, \mathbf{0}, rule\rangle + \langle 0, c_f \cdot w_g, \mathbf{g}, rule\rangle + \\ &\quad \langle 0, c_g \cdot w_f, \mathbf{f}, rule\rangle + \langle 0, w_f \cdot w_g, \mathbf{f}\cdot\mathbf{g}, rule\rangle \\ &= \langle c_h, w_h, \mathbf{h}, rule_h\rangle\end{aligned}$$

We now have only $O(|\langle c_f, w_f, \mathbf{f}, rule_f\rangle| \cdot |\langle c_g, w_g, \mathbf{g}, rule_g\rangle|)$ calls to *multiply* but every call requires three calls to *apply* for adding the separate terms. The first addition is not costly since the first term is always a constant, however, the second and third addition are potentially costly.

In addition to the *additive property* two further properties, the *bounding property* and the *domain-reducing property*, have been introduced by Lai, et al. [16] [14]. As has been done for the *additive property*, these properties can be adapted to FEVBDDs.

Definition 3.5 *An operator op applied to* $\langle c_f, w_f, \mathbf{f}, rule_f\rangle$ *and* $\langle c_g, w_g, \mathbf{g}, rule_g\rangle$ *is said to satisfy the bounding property if*

$$((c_f + w_f \cdot m(f))op(c_g + w_g \cdot m(g))) = 0,\ 1,\ (c_f + w_f \cdot f),\ or\ (c_g + w_g \cdot g),$$

where $m(f)$ $(m(g))$ *is the maximum or minimum of* f (g).

In order to take advantage of the *bounding property* we have to store both the minimum and maximum value of the function f represented by the FEVBDD node $\mathbf{v}$ with the node. As a consequence of the *bounding property*, we can immediately determine the result of an operation without further computation if the maximum or minimum of a function exceeds a boundary value. Furthermore, storing the min and max values with every node enables us to easily incorporate branch-and-bound algorithms that efficiently solve combinatorial optimization problems. As an example, we present an algorithm $leq0(\langle c_f, w_f, \mathbf{f}, rule_f\rangle)$ that performs the operation $(c_f + w_f \cdot f) \leq 0)$ in table 3.4.

Definition 3.6 *Given a FEVBDD $\langle c_f, w_f, \mathbf{f}, rule_f\rangle$, the domain of a node $\mathbf{v} \in \mathbf{f}$ with respect to an operator op is defined as:*

$$D^{op}_{\mathbf{v}} = \{(c_u, w_u) | \langle c_u, w_u, \mathbf{u}, rule_u\rangle \textit{are the nodes that need to be generated with respect to op}\}$$

Definition 3.7 *An operator op applied to $\langle c_f, w_f, \mathbf{f}, rule_f\rangle$ and $\langle c_g, w_g, \mathbf{g}, rule_g\rangle$ is said to satisfy the domain-reducing property if there exists some node $\mathbf{v} \in \mathbf{f}$ or $\mathbf{v} \in \mathbf{g}$ such that $D^{op}_{\mathbf{v}} \subset D^{eval}_{\mathbf{v}}$.*

The *modulo* operator mod, for example, satisfies the *domain reducing-property* since it holds that:

$$(c_f + w_f \cdot f) \bmod c = ((c_f \bmod c) + (w_f \bmod c) \cdot f) \bmod c \tag{3.1}$$

Equation 3.1 holds since it can be easily proven that the following equations are valid.

$$\begin{aligned}
(a \cdot b) \bmod c &= ((a \bmod c) \cdot (b \bmod c)) \bmod c \\
&= ((a \bmod c) \cdot b) \bmod c \\
&= (a \cdot (b \bmod c)) \bmod c \\
(a + b) \bmod c &= ((a \bmod c) + (b \bmod c)) \bmod c \\
&= ((a \bmod c) + b) \bmod c \\
&= (a + (b \bmod c)) \bmod c
\end{aligned}$$

Thus, the domain of a node $\mathbf{v} \in \langle c_f, w_f, \mathbf{f}, rule_f\rangle$ is $D^{\mathrm{mod}}_{\mathbf{v}} = D^{eval}_{\mathbf{v}} \cap \{0, \ldots, c-1\} \times \{0, \ldots, c-1\}$. Therefore, $\langle c_f + k_1 \cdot c, k_2 \cdot c \cdot w_f, \mathbf{f}, rule\rangle$ can share the computation result of $\langle c_f, w_f, \mathbf{f}, rule_f\rangle$ for arbitrary integers k_1 and k_2.

```
leq0(⟨c_f, w_f, f, rule_f⟩)
    if(w_f < 0) {
        max = c_f + w_f · min_f;
        min = c_f + w_f · max_f;
    }
    else {
        max = c_f + w_f · max_f;
        min = c_f + w_f · min_f;
    }

    if(max ≤ 0)
        return(⟨1, 0, 0, rule⟩);
    if(min > 0)
        return(⟨0, 0, 0, rule⟩);
    if(comp_table_lookup(⟨c_f, w_f, f, rule_f⟩,leq0,ans)
        return(ans);

    ⟨c_{h_t}, w_{h_t}, h_t, rule_{h_t}⟩ = leq0(⟨c_f + w_f · ev, w_f · w_{t_f}, child_t(f), rule⟩);
    ⟨c_{h_e}, w_{h_e}, h_e, rule_{h_e}⟩ = leq0(⟨c_f, w_f · w_{e_f}, child_e(f), rule⟩);

    if(⟨c_{h_t}, w_{h_t}, h_t, rule_{h_t}⟩ == ⟨c_{h_e}, w_{h_e}, h_e, rule_{h_e}⟩)
        return(⟨c_{h_t}, w_{h_t}, h_t, rule_{h_t}⟩);

    ⟨c_h, w_h, h, rule_h⟩=make_new_node(variable(f),h_t,h_e);
    comp_table_insert(⟨c_f, w_f, f, rule_f⟩,leq0,⟨c_h, w_h, h, rule_h⟩);
    return(⟨c_h, w_h, h, rule_h⟩ );
}
```

Table 3.4: *less-or-equal-zero leq0*

3.2 Representation of Boolean Functions

Boolean Functions are represented in FEVBDDs by encoding the boolean values *true* and *false* as integers 1 and 0, respectively. All the basic boolean operations can be easily represented using only arithmetic operations (cf. table 2.3). Thus we can easily represent any boolean function using FEVBDDs. Although we could implement the boolean operations based on their corresponding arithmetic functions, it is by far better in terms of computational complexity to directly use *apply* for boolean operations. All we need to do is to provide the necessary terminal cases for *apply*($\langle c_f, w_f, \mathbf{f}, rule_f\rangle$,$\langle c_g, w_g, \mathbf{g}, rule_g\rangle$,*boolean_op*). In the case of the boolean conjunction operation for example the terminal cases are:

1. if($\langle c_f, w_f, \mathbf{f}, rule_f\rangle = \langle 0, 0, \mathbf{0}, rule\rangle$ or $\langle c_g, w_g, \mathbf{g}, rule_g\rangle = \langle 0, 0, \mathbf{0}, rule\rangle$) return $\langle 0, 0, \mathbf{0}, rule\rangle$

2. if($\langle c_f, w_f, \mathbf{f}, rule_f\rangle = \langle 1, 0, \mathbf{0}, rule\rangle$) return $\langle c_g, w_g, \mathbf{g}, rule_g\rangle$

3. if($\langle c_g, w_g, \mathbf{g}, rule_g\rangle = \langle 1, 0, \mathbf{0}, rule\rangle$) return $\langle c_f, w_f, \mathbf{f}, rule_f\rangle$

To convert a boolean function from its OBDD to its FEVBDD representation we can adapt the algorithm suggested by Lai in [16]. Additionally, the concept of multiplicative weights allows us to directly represent the so called complement edges, so that we need to take care of this case in the algorithm.

1. convert the terminal node **0** to $\langle 0, 0, \mathbf{0}, rule\rangle$ and **1** to $\langle 1, 0, \mathbf{0}, rule\rangle$.

2. for each nonterminal node $\langle x_i, \mathbf{t}, \mathbf{e}\rangle$ in the OBDD such that **t** and **e** have already been converted to FEVBDDs as $\langle c_{t'}, w_{t'}, \mathbf{t}', rule_{t'}\rangle$ and $\langle c_{e'}, w_{e'}, \mathbf{e}', rule_{e'}\rangle$, the following conversion rules are applied:

3. if the branch leading from node $\langle x_i, \mathbf{t}, \mathbf{e}\rangle$ to **t** or **e** is a complement edge we have to perform the complementation by computing $1 - t$ or $1 - e$, respectively. This is achieved

by multiplying both weights c_t (c_e) and w_t (w_e) by -1 and later adding 1 to c_t (c_e). The four basic conversion rules are listed below:

- $\langle x_i, \langle 0, 1, \mathbf{t}', rule\rangle, \langle 0, 1, \mathbf{e}', rule\rangle\rangle \rightarrow \langle 0, 1, \langle x_i, \mathbf{t}', \mathbf{e}', 0, 1, 1\rangle\rangle$
- $\langle x_i, \langle 0, 1, \mathbf{t}', rule\rangle, \langle 1, 1, \mathbf{e}', rule\rangle\rangle \rightarrow \langle 1, 1, \langle x_i, \mathbf{t}', \mathbf{e}', -1, 1, 1\rangle\rangle$
- $\langle x_i, \langle 1, 1, \mathbf{t}', rule\rangle, \langle 0, 1, \mathbf{e}', rule\rangle\rangle \rightarrow \langle 0, 1, \langle x_i, \mathbf{t}', \mathbf{e}', 1, 1, 1\rangle\rangle$
- $\langle x_i, \langle 1, 1, \mathbf{t}', rule\rangle, \langle 1, 1, \mathbf{e}', rule\rangle\rangle \rightarrow \langle 1, 1, \langle x_i, \mathbf{t}', \mathbf{e}', 0, 1, 1\rangle\rangle$

The above conversion rules are not complete in the case of FEVBDDs since we can now also have variations in the multiplicative weights which can either be +1 or −1. These cases however are handled exactly according to the norm weighting rule that has been presented before, so that we do not explicitly list them here.

As it has been done for EVBDDs [16], it can be shown that the following theorems hold.

Theorem 3.2 *Given an OBDD representation* $\mathbf{v}$ *of a boolean function with complement edges being allowed and an FEVBDD* $\langle c_v, w_v, \mathbf{v}, rule_v\rangle$, *then* $\mathbf{v}$ *and* $\mathbf{v}'$ *have the same topology except that the terminal node* **1** *is absent from the FEVBDD* $\mathbf{v}'$ *and the edges connected to it are redirected to the terminal node* **0**.

Theorem 3.3 *Given two OBDDs* $\mathbf{f}$ *and* $\mathbf{g}$ *with complement edges being allowed and the corresponding FEVBDDs* $\langle c_{f'}, w_{f'}, \mathbf{f}', rule_{f'}\rangle$ *and* $\langle c_{g'}, w_{g'}, \mathbf{g}', rule_{g'}\rangle$, *the time complexity of boolean operations on FEVBDDs (using apply) is* $O(|\mathbf{f}| \cdot |\mathbf{g}|) = O(|\mathbf{f}'| \cdot |\mathbf{g}'|)$.

In table 2.5 the complexity of boolean operations performed on both the FEVBDD and *BMD representations of the boolean functions is compared. It can be seen, that the complexity for FEVBDDs is significantly lower than the complexity for *BMDs.

An example of a FEVBDD representing a boolean function with complement edges is given in figure 3.1. The FEVBDD of figure 3.1 represents the four output functions of a 3-bit adder.

It has the same topology (except for the terminal edges) as the corresponding OBDD which can be found in figure 2.2. As it is shown in this example, FEVBDDs successfully extend the use of EVBDDs to represent boolean functions as they inherently offer a way to represent complement edges. Furthermore, the boolean operation '*not*' can now be performed in constant time since it only requires manipulation of the weights of the root node.

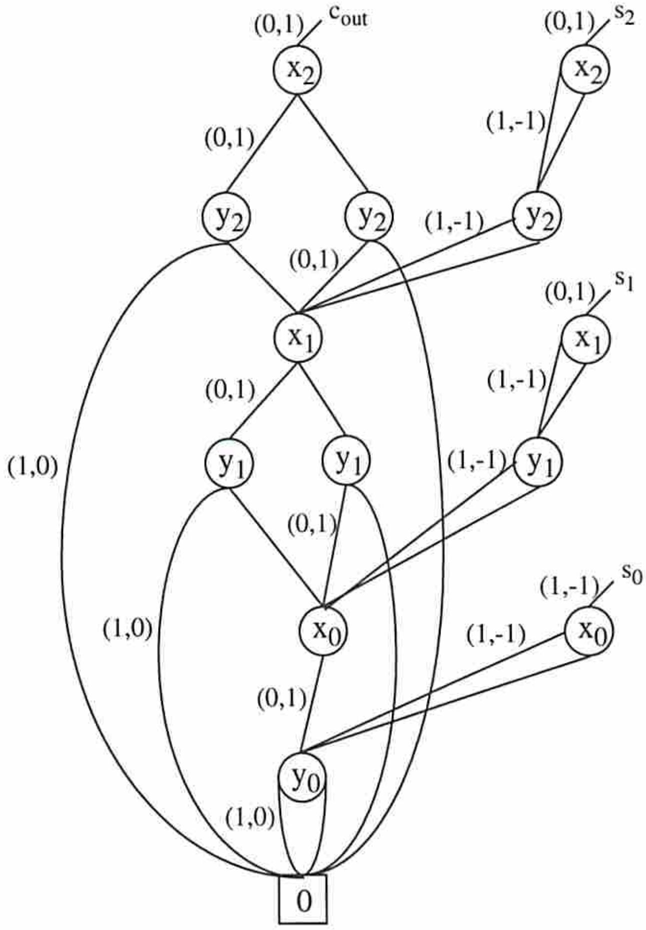

Figure 3.1: FEVBDD representation of the four output functions of a 3-bit adder

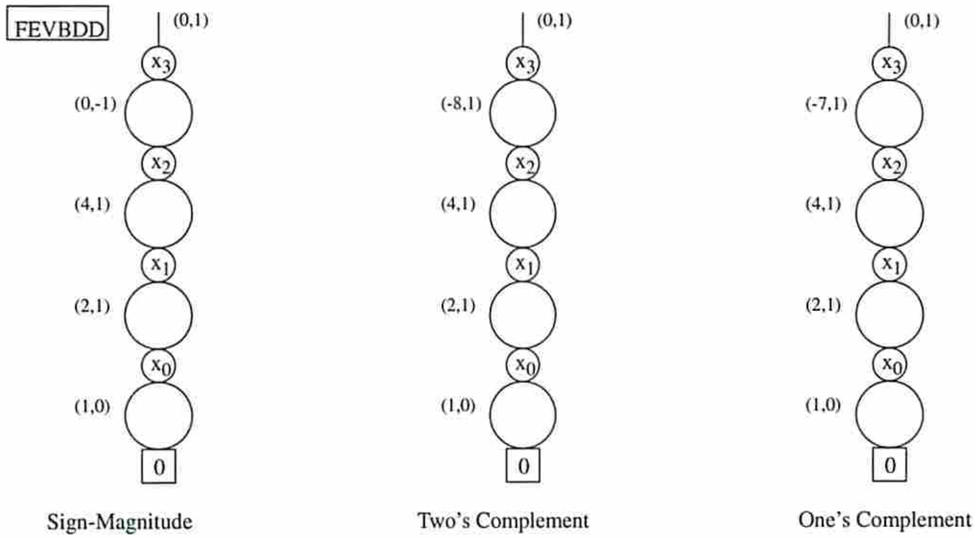

Figure 3.2: Representations of signed integers using FEVBDDs

3.3 Logic Verification

The purpose of logic verification is to formally prove that the actual implementation satisfies the conditions defined by the specification. This is done by formally showing the equivalence between the combinational circuit, i.e. the description of the design and the specification of the intended behaviour.

In general, the implementation is represented by an array of boolean functions $\mathbf{f}_b$ and the specification is given by a word-level function f_w. In order to transform the bit-level representation to the word-level we can use any encoding function to encode the binary input signals to the circuit. The set of input signals is partitioned into several subsets of binary signals $\mathbf{x}^0, \ldots, \mathbf{x}^n$ and every array $\mathbf{x}^i$ is then encoded using an encoding function $encode_i$ that provides a word-level interpretation of the binary input signals. Common encoding functions are signed-integer, one's-complement and two's-complement. The corresponding FEVBDDs are shown in figure 3.2. Thus, the implementation can be described by an array of boolean functions $\mathbf{f}_b(\mathbf{x}^0, \ldots, \mathbf{x}^n)$. The specification is given as a word-level function $f_w(X_1, \ldots, X_n)$.

	X	$X+Y$	$X \cdot Y$	X^2	c^X
MTBDD	exponential	exponential	exponential	exponential	exponential
EVBDD, FEVBDD	linear	linear	exponential	exponential	exponential
BMD	linear	linear	quadratic	quadratic	exponential
*BMD	linear	linear	linear	quadratic	linear

Table 3.5: Complexity of Representing Word-Level Operations

Verification is then done by proving the equivalence between an encoding of the binary output signals of the circuit, i.e. the array of boolean functions, and the word-level function of the encoded input signals:

$$encode_{out}(\mathbf{f}_b(\mathbf{x}^0, \ldots, \mathbf{x}^n)) = f_w(encode_0(\mathbf{x}^0), \ldots, encode_n(\mathbf{x}^n))$$

This strategy for logic verification was first proposed by Lai, et al., using EVBDDs [14]. Since FEVBDDs can describe both bit-level and word-level functions, they can be successfully applied to logic verification.

Although all word-level operations can be represented by FEVBDDs, the space complexity of certain operations becomes exponential so that their application is limited to small word-length. As can be seen in table 3.5 this is a major disadvantage compared with *BMDs. Whereas the size of the FEVBDD representing word-level multiplication grows exponentially the *BMD size grows linearly. The FEVBDD, EVBDD and *BMD representation of word-level addition and multiplication are given in figures 3.3 and 3.4, respectively. Word-level addition can be represented very efficiently by all three function graphs.

Though both EVBDD and FEVBDD representations of word-level multiplication are exponential, FEVBDDs offer significant savings in memory consumption over EVBDDs. As can be seen in figure 3.4 for word-level multiplication of two three-bit integers, the EVBDD contains 28 internal nodes whereas the FEVBDD representation requires only 10 nodes. In general, the EVBDD denoting the multiplication of two n-bit integers has $(n+1)(2^n-1)$ internal

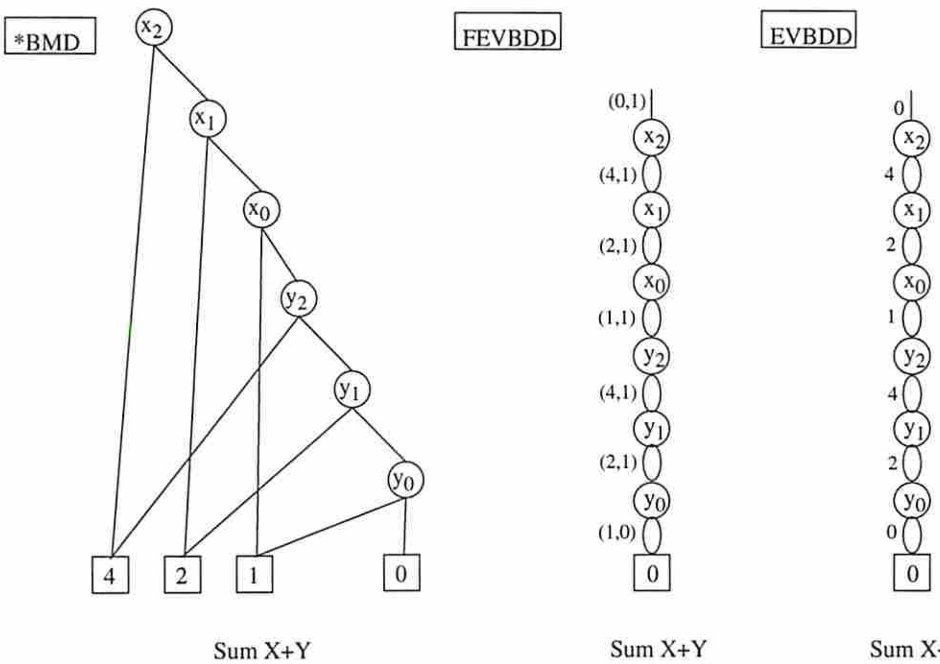

Figure 3.3: Representations of word-level sum

nodes. The corresponding FEVBDD contains only $n + (2^n - 1)$ internal nodes and the ratio of EVBDD nodes to FEVBDD nodes is $\frac{n+1}{1+\frac{n}{2^n-1}}$. As can be seen from this ratio, the savings in the number of nodes in the FEVBDD representation are of order n. As an example, a 16-bit multiplier requires 1,114,095 EVBDD nodes but only 65,551 FEVBDD nodes. Even if we take into account that a FEVBDD node requires 20 bytes versus only 12 bytes per EVBDD node, the savings remain significant (EVBDD:13.3 Mbyte, FEVBDD: 1.3 Mbyte).

Although *BMDs are superior to FEVBDDs when representing word-level operations, they suffer from shortcomings in the computational complexity for boolean operations as has been discussed in section 2.6. Thus, FEVBDDs may be more efficient to use than *BMDs if the specification contains mostly add, subtract and scalar multiplication operations and requires only word-level multiplication of short integers. Handling world-level multiplication of up to 16-bit integers appears to be feasible.

As has been done for EVBDDs, FEVBDDs can also be extended to structured FEVBDDs

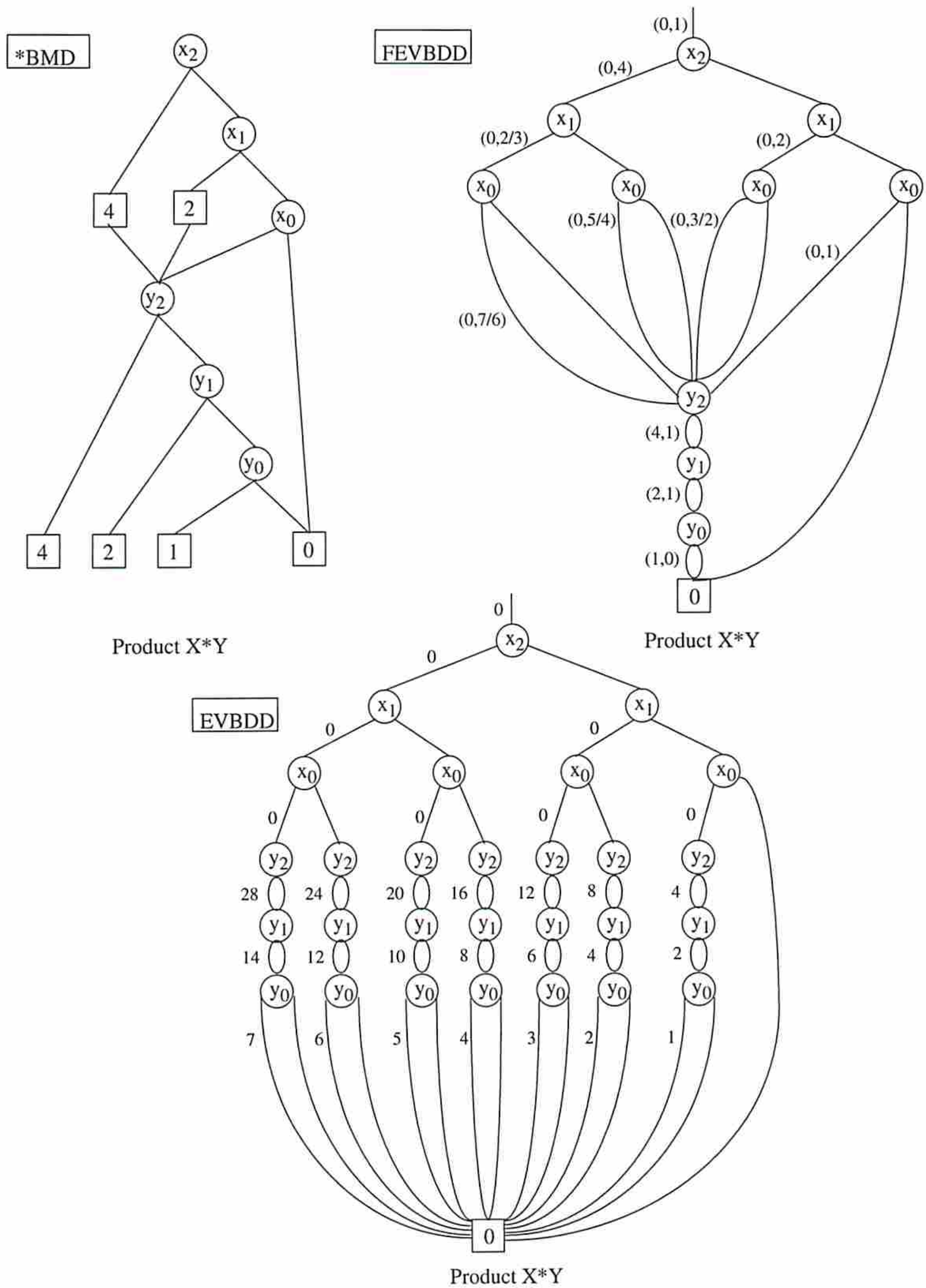

Figure 3.4: Representations of word-level product

which allow the modeling of conditional expressions and vectors.

3.4 Integer Linear Programming

An algorithm FGILP for solving Integer Linear Programming (ILP) problems based on EVBDDs has been proposed by Lai, et al., in [17]. FGILP realizes an ILP solver based on function graphs, which uses a mixed branch-and-bound/implicit-enumeration strategy. It has been shown that this approach can successfully compete with other branch-and-bound strategies that require the solution of the corresponding Linear Programming problems. The latter strategy is the one most widely applied in commercial programs.

An ILP problem can be formulated as follows:

$$\textit{minimize} \quad \sum_{i=1}^{n} c_i x_i \tag{3.2}$$

$$\text{subject to} \quad \sum_{i=1}^{n} a_{ij} x_i \leq b_j, 1 \leq j \leq m, \tag{3.3}$$

$$\text{with } x_i \text{ integer}$$

Since both EVBDDs and FEVBDDs allow only binary decision variables, the encodings shown in figure 3.2 have to be applied. A 32-bit integer, for example, can be represented by an EVBDD or FEVBDD with 32 nodes. Since FEVBDDs form an extension of EVBDDs we can also apply FEVBDDs to solve ILP problems. We expect a reduction in the memory requirement for FGILP when using FEVBDDs. This is due to the fact that different multiples of the integer variables x_i appear in equations (3.2) and (3.3). If we use EVBDDs to represent these multiples of x_i, we have to build an EVBDD for every different coefficient a_{ij} since scalar multiplication on EVBDDs is performed by multiplying all edge weights with the factor. If we use FEVBDDs, however, we only have to store the FEVBDD representing x_i once. Multiples of x_i can be easily realized by associating the corresponding multiplicative edge weights with dangling incoming edges leading to x_i. As an example, storing $6x$, $7x$ and $5x$ requires 96 nodes if we use EVBDDs but only 32 nodes if we apply FEVBDDs.

3.5 Implementation of Arbitrary Precision Arithmetic

The introduction of multiplicative weights in combination with the *RATIONAL rule* for weight normalizing makes it necessary to extend the value range of the edge weights from the integer domain to the rational domain. This is done in a way such that any future expansion to other domains such as the complex domain can be easily achieved. All operations on edge weights are accessed through a standardized interface that invokes the specified function and then executes the requested operation depending on the current mode. Thus, the FEVBDD code remains fully independent of the selected domain. By changing to another mode we can easily switch from the integer domain to the rational domain, for example. This means we can still use the fast routines for single precision integers when necessary.

Multiple precision integers are realized as arrays of integers and the arithmetic operations are implemented based on the algorithms for multiple precision arithmetic given by Knuth in [13]. Multiple precision fractions are implemented as arrays of two multiple precision integers where one integer represents the numerator and the other one the denominator. It is enforced by the package that the numerator and denominator remain relative prime and only the numerator can be signed. This is achieved by computing the greatest common divisor (GCD) of numerator and denominator and dividing both the numerator and denominator by the GCD. This operation is performed whenever an input is given. Internally the data is guaranteed to remain in the normalized form as this form is strictly enforced by all operations. Thus, a rational value is always uniquely represented by the numerator and denominator.

The GCD can be computed very fast by Euclid's algorithm or the binary GCD algorithm [13]. For multi-precision fractions we use the binary GCD algorithm since it works very fast for integers of multiple word length. It only relies on subtraction and right shifting and does not require division operations. For single word precision fractions we employ the classical version of Euclid's algorithm since division can be executed very efficiently for single word integers. The basic arithmetic operations for fractions are realized as follows:

- multiplication:

$$U \cdot V = \frac{u}{u'} \cdot \frac{v}{v'} = \frac{\frac{u}{d_1} \cdot \frac{v}{d_2}}{\frac{u'}{d_2} \cdot \frac{v'}{d_1}}$$

  where $d_1 = \gcd(u, v')$ and $d_2 = \gcd(u', v)$

- division:

$$\frac{U}{V} = \frac{\frac{u}{u'}}{\frac{v}{v'}} = \frac{u}{u'} \cdot \frac{v'}{v} = \frac{\frac{u}{d_1} \cdot \frac{v'}{d_2}}{\frac{u'}{d_2} \cdot \frac{v}{d_1}}$$

  where $d_1 = \gcd(u, v)$ and $d_2 = \gcd(u', v')$

- addition:

$$U + V = \frac{u}{u'} + \frac{v}{v'} = \begin{cases} \frac{u \cdot v' + v \cdot u'}{u' \cdot v'} & \text{if } d_1 = 1 \\ \frac{\frac{t}{d_2}}{\frac{u'}{d_1} \cdot \frac{v'}{d_2}} & \text{if } d_1 > 1 \end{cases}$$

  where $d_1 = \gcd(u', v')$ and if $d_1 > 1$: $t = u \cdot \frac{v'}{d_1} + v \cdot \frac{u'}{d_1}$, $d_2 = \gcd(t, d_1)$

Chapter 4

Matrix Representation and Manipulation

Matrices have been successfully represented using MTBDDs [9] [10] and ADDs [2] and implementations of the basic matrix operations such as addition and multiplication have been given. A popular class of matrices that can be efficiently represented by MTBDDs and EVBDDs is the class of Walsh matrices which can be generated by a recursive rule.

4.1 Representation of Matrices

The basic idea in using function graphs to represent matrices is to encode both the row and column position of the matrix elements using binary variables. An 8×8 matrix, for example, requires 3 binary variables for the rows and another 3 for the columns. Basically, we can view the problem of representing a $m \times n$ matrix as representing a function from the finite set $D = \{0, \ldots, m-1\} \times \{0, \ldots, n-1\}$ of all element positions to the finite set R of its elements. The binary variables giving the row position are called *row designators* $x \in \{x_0, \ldots, x_{m-1}\}$, the ones denoting the column position are called *column designators* $y \in \{y_0, \ldots, y_{n-1}\}$. For the imposed variable ordering row and column designators are mixed together such that the order is $\{x_0, y_0, \ldots, x_{m-1}, y_{n-1}\}$. Because of this chosen variable ordering subtrees in the function graph directly correspond to submatrices in the given matrix, as can be seen in figure 4.1.

Based on this correspondence the pseudo-boolean function denoting the matrix M can be given easily:

$$\begin{array}{rcl} f_M & = & f_{M_{\overline{xy}}}\ (1-x)\ (1-y)\ +\ f_{M_{\bar{x}y}}\ (1-x)\ y\ + \\ & & f_{M_{x\bar{y}}}\ x\ (1-y)\ +\ f_{M_{xy}}\ x\ y \end{array}$$

Furthermore, this ordering allows matrices to be be represented compactly if they have sub-

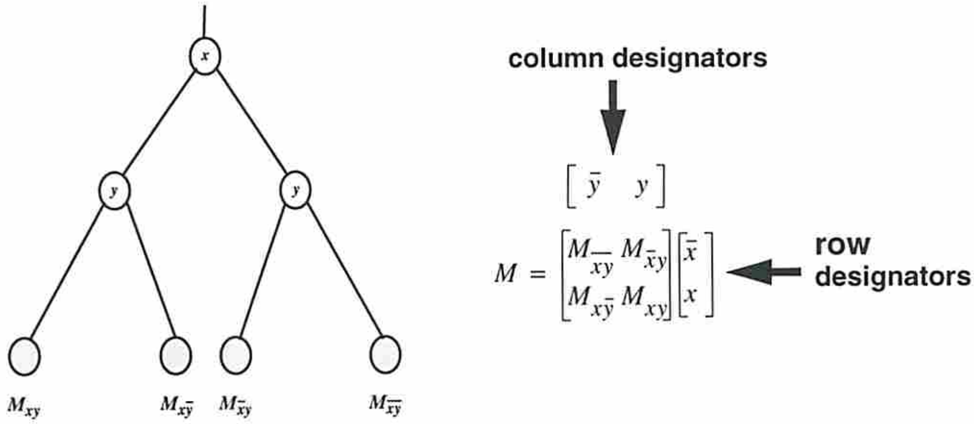

Figure 4.1: Correspondence of submatrices and subtrees

matrices that are identical (MTBDDs) or can be transformed into each other by an affine transformation[1] (FEVBDDs). Since the concept of square matrices, i.e. vertical size m = horizontal size n, helps to keep many algorithms efficient and simple we will from now on only consider square matrices with $size = \max(m, n)$. To make non-square matrices square we can easily pad them with rows or columns filled with zeros. This does not significantly increase our memory consumption for storing the matrix since the padded blocks are uniform and can therefore be represented by only a few nodes.

As it has already been mentioned, MTBDDs only offer a compact and memory efficient representation of matrices that feature identical subblocks. They require a different terminal node for each distinct matrix element. FEVBDDs can do far better than that. The concept of FEVBDDs allows two subblocks to be represented by the same subgraph if they differ only by

[1] An affine transformation is a transformation of the form $y \rightarrow a \cdot y + b$.

an affine transformation of their elements. We will now introduce a special class of matrices that can always be represented by a FEVBDD of linear size. For this class of matrices the sizes of the corresponding MTBDD, EVBDD and *BMD are likely to be exponential.

Definition 4.1 *A recursively-affine matrix is recursively generated using the following rules:*

1. *we begin with a* 1×1 *matrix* $M_0 = [c_0]$ *where* c_0 *is a integer or rational constant value*

2. *in every recursion step a new matrix* M_{n+1} *is created based on the previous result* M_n *such that:*
$$M_{n+1} = \begin{bmatrix} k_0 + w_0 \cdot M_n & k_1 + w_1 \cdot M_n \\ k_2 + w_2 \cdot M_n & k_3 + w_3 \cdot M_n \end{bmatrix}$$
with $k_0, \ldots, k_3$ *and* $w_0, \ldots, w_3$ *being arbitrary integer or rational numbers.*

Figure 4.2 shows the general structure of the FEVBDD that corresponds to a recursion step in building up a recursively affine matrix. In every recursion step a structure as shown in figure 4.2 is added to the already constructed FEVBDD. As can be seen from figure 4.2 we only need $3 \cdot \lceil \log_2(n) \rceil$ nodes to represent a recursively-affine matrix of size n x n. As an example of a recursively affine matrix we build the FEVBDD for the matrix M given below:

$$M = \begin{bmatrix} 3 & 10 & 14 & 35 \\ 9 & 5 & 32 & 20 \\ 12 & 26 & 22 & 64 \\ 24 & 16 & 58 & 34 \end{bmatrix}$$

In order to illustrate the potential advantages of FEVBDDs over *BMDs in matrix applications, we also build the *BMD representation of M as shown in figure 4.3. An important class of matrices that belongs to the family of recursively-affine matrices is the set of Walsh matrices in the Hadamard ordering [21]. These matrices can be used to compute spectral transforms of boolean functions. They are recursively defined as follows:

$$\begin{aligned} H_1^h &= [1] \\ H_{n+1}^h &= \begin{bmatrix} H_n^h & H_n^h \\ H_n^h & -H_n^h \end{bmatrix} \end{aligned}$$

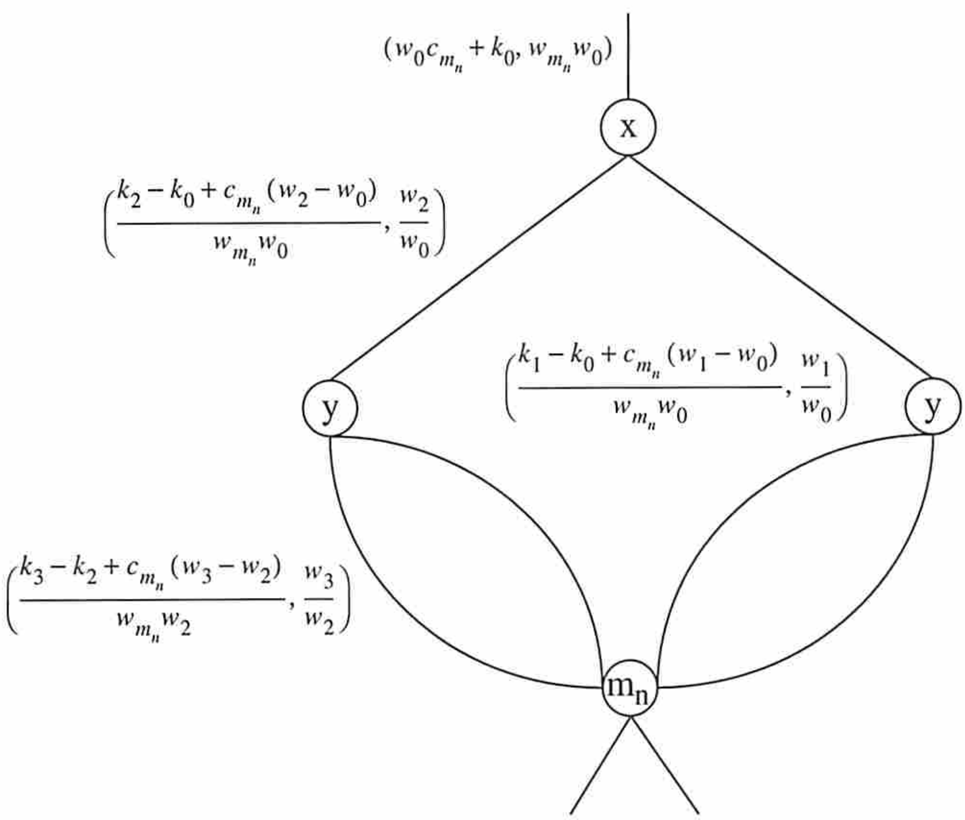

Figure 4.2: Recursively-affine matrices

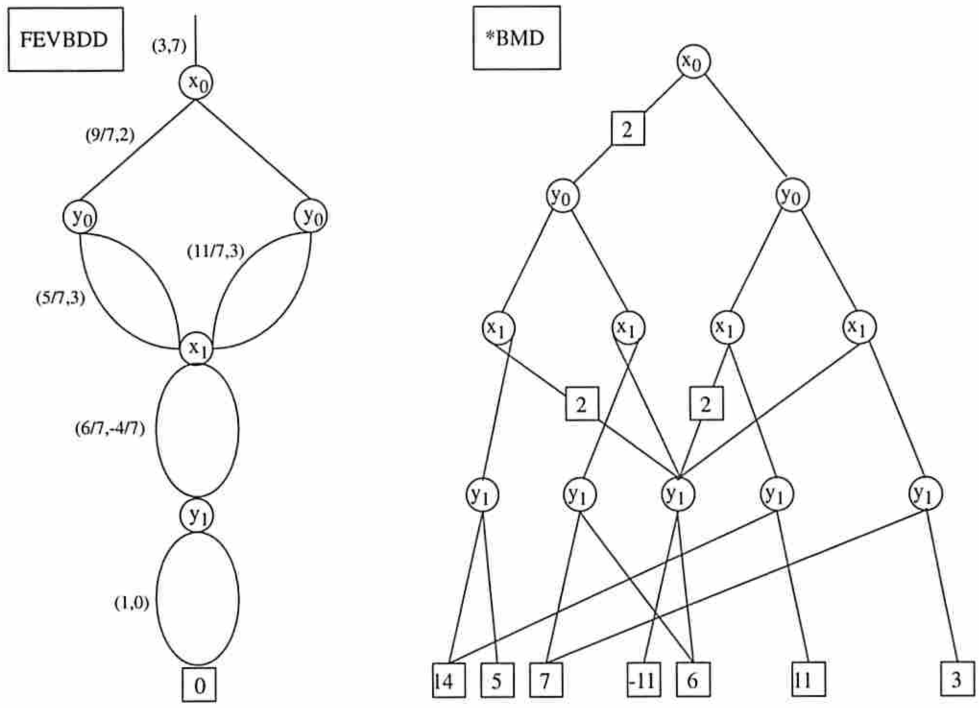

Figure 4.3: FEVBDD and *BMD representations of a recursively affine matrix M

Figure 4.4 shows both the FEVBDD and EVBDD representations of the Walsh matrix H_3^h. As can be seen in figure 4.4, the size of the FEVBDD representation is $2 \cdot n$ where n denotes the order of the Walsh matrix. The size of the EVBDD representation is $4 \cdot n - 2$

Generally speaking, employing function graphs such as MTBDDs or FEVBDDs to represent sparse matrices offers the following advantages:

1. In comparison with normal sparse data structures, function graphs do provide a uniform $\log_2(N)$ access time, where N is the number of real elements being stored in the function graph (for example, all non-zero elements of a sparse matrix)

2. Function graphs may not be able to beat sparse-matrix data structures in terms of worst space complexity. However, recombination of isomorphic subgraphs may give a considerable practical advantage to function graphs over other data structures. This is particularly valid for FEVBDDs since the same subgraph can represent all the matrices

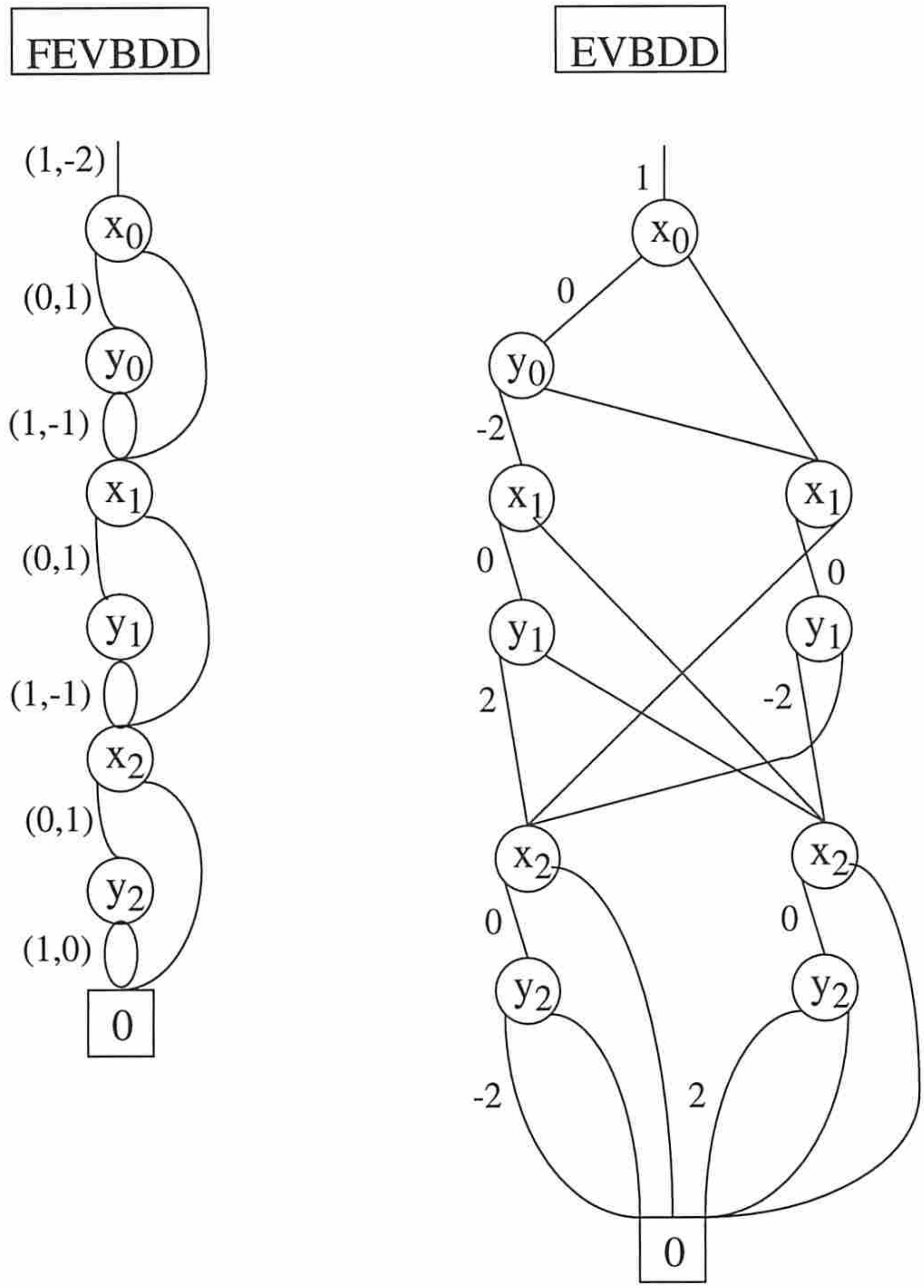

Figure 4.4: FEVBDD and EVBDD representations of the Walsh matrix H_3^h

that can be generated by an affine transformation of the matrix represented by the subgraph.

4.2 Operations

Operations on matrices can be divided into two major groups. The first group comprises termwise operations such as scalar multiplication, addition, etc. The second group is formed by matrix multiplication, matrix transpose and matrix inversion. Termwise operations are easily implemented based on function graphs. We can simply use *apply* to compute all termwise operations on matrices. This is obviously possible since *apply*(*op*) performs the operation *op* on every single function value, i.e. it works in a termwise manner. Matrix specific operations such as transposition require their own tailored algorithms.

Matrix multiplication is clearly a non-termwise operation since it requires computing the scalar vector product of a row of the left matrix with a column of the right matrix to get the value of a single matrix element of the product matrix. Therefore, we will present two different recursive procedures to perform matrix multiplication on function graphs. The first method is called the 'Berkeley-method' and was proposed by McGeer [10]. This algorithm has the most direct link to the common conventional method for matrix multiplication. In every recursion step the problem is divided into four subproblems until a terminal case has been reached. In these steps operands are expanded with regard to a pair of row and column designators. This expansion even takes place if the function graphs are actually not dependent on the current pair of internal variables. By doing so there is no need for a scaling step as is necessary in the second method. Let matrix multiplication be denoted by $\star$ and matrix addition by $+$. This method can be formally stated as:

$$h(\{x_0, y_0, \ldots, x_{m-1}, y_{n-1}\}) = f(\{x_0, z_0, \ldots, x_{m-1}, z_{p-1}\}) \star g(\{z_0, y_0, \ldots, z_{p-1}, y_{n-1}\})$$

or written in terms of matrices:

$$\begin{bmatrix} h_{\overline{x}\overline{y}} & h_{\overline{x}y} \\ h_{x\overline{y}} & h_{xy} \end{bmatrix} = \begin{bmatrix} f_{\overline{x}\overline{z}} & f_{\overline{x}z} \\ f_{x\overline{z}} & f_{xz} \end{bmatrix} \star \begin{bmatrix} g_{\overline{z}\overline{y}} & g_{\overline{z}y} \\ g_{z\overline{y}} & g_{zy} \end{bmatrix}$$

The computations performed in every recursion step are:

$$\begin{aligned} h_{\overline{x}\overline{y}} &= f_{\overline{x}\overline{z}} \star g_{\overline{z}\overline{y}} + f_{\overline{x}z} \star g_{z\overline{y}} \\ h_{\overline{x}y} &= f_{\overline{x}\overline{z}} \star g_{\overline{z}y} + f_{\overline{x}z} \star g_{zy} \\ h_{x\overline{y}} &= f_{x\overline{z}} \star g_{\overline{z}\overline{y}} + f_{xz} \star g_{z\overline{y}} \\ h_{xy} &= f_{x\overline{z}} \star g_{\overline{z}y} + f_{xz} \star g_{zy} \end{aligned}$$

Obviously, this method requires eight calls to *matrix multiply* and four calls to *matrix add* in every recursion step, i.e. for every internal variable pair.

The second method called the 'Boulder method' was proposed by Bahar [2]. Unlike the previous method it only expands the top variable of the two operands $f\{x_0, z_0, \ldots, x_{n-1}, z_{n-1}\}$ and $g\{z_0, y_0, \ldots, z_{n-1}, y_{n-1}\}$. In the process of matrix multiplication, the following variable order $\{x_0, z_0, y_0, \ldots, x_{n-1}, z_{n-1}, y_{n-1}\}$ is imposed to decide whether the top variable of f or g has to be selected as the top variable for expansion. Depending on the character of the expansion variable var one of the following computations is being made in every recursion step.

- if($var = z_i$) then

$$f(x,z) \star g(z,y) = (f_v(x,z) \star g_v(z,y)) + (f_{\overline{v}}(x,z) \star g_{\overline{v}}(z,y))$$

- if($var = x_i$ or $var = y_i$) then

$$f(x,z) \star g(z,y) = v \wedge (f_v(x,z) \star g_v(z,y)) \vee \overline{v} \wedge (f_{\overline{v}}(x,z) \star g_{\overline{v}}(z,y))$$

This approach only expands internal variables that are actually encountered in the function graphs f and g. It requires to keep track of missing z variables in f and g since every z expansion step corresponds to performing matrix addition. If p gives the number of omitted z expansions between two recursion steps we have to scale the result by 2^p before returning it. When using a cache we always store the unscaled results and scale the entry accordingly when reading the cache.

Another method called the CMU method was proposed by Clarke [10]. Its basic idea is to take all the products first and then compute all the sums.

For our matrix package we have implemented the second method which has proven superior to the other two [2]. We implemented two different versions of this method. Version 1 passes the value of the edge weights down with every recursion step of *matrix multiply* and is of $O(\|f\| \cdot \|g\|)$ complexity. As we have done for multiplication of two FEVBDDs we suggest a second version for function graphs with a high ratio of reconverging branches (e.g. for recursively-affine matrices) as follows.

$$\begin{aligned}
\langle c_h, w_h, \mathbf{h}, rule_h\rangle &= \langle c_f, w_f, \mathbf{f}, rule_f\rangle \star \langle c_g, w_g, \mathbf{g}, rule_g\rangle \\
&= ([c_f]_{n\times n} + w_f \cdot [f]_{n\times n}) \star ([c_h]_{n\times n} + w_h \cdot [h]_{n\times n}) \\
&= [2^n \cdot c_f \cdot c_g]_{n\times n} + [c_f \cdot w_g]_{n\times n} \star [h]_{n\times n} + \\
&\quad [f]_{n\times n} \star [c_g \cdot w_f]_{n\times n} + [f]_{n\times n} \star [g]_{n\times n} \\
&= [2^n \cdot c_f \cdot c_g]_{n\times n} + c_f \cdot w_g \cdot coladd([h]_{n\times n}) + \\
&\quad c_g \cdot w_f \cdot rowadd([f]_{n\times n}) + [f]_{n\times n} \star [g]_{n\times n}
\end{aligned}$$

The operations *rowadd* and *coladd* which generate matrices such that

$$rowadd(\begin{bmatrix} a_{00} & \dots & a_{0n} \\ \vdots & & \vdots \\ a_{n0} & \dots & a_{nn} \end{bmatrix}) = \begin{bmatrix} \sum_i a_{0i} \\ \vdots \\ \sum_i a_{ni} \end{bmatrix}$$

$$coladd(\begin{bmatrix} a_{00} & \dots & a_{0n} \\ \vdots & & \vdots \\ a_{n0} & \dots & a_{nn} \end{bmatrix}) = \begin{bmatrix} \sum_i a_{i0} & \dots & \sum_i a_{in} \end{bmatrix}$$

only have complexity $O(|f|)$. This second version requires only $O(|f| \cdot |g|)$ calls to *matrix multiply* but every recursive call to *matrix multiply* also requires three calls to *matrix add.* It improves the cache efficiency of matrix multiplication considerably, if both operands are represented by FEVBDDs with a high ratio of reconverging branches. This outweighs the added overhead of three calls to *matrix add.* If this is not the case, it is better to use the original approach since it does not require the additional overhead.

Matrix transposition is performed by exchanging the roles of column and row designators belonging to the same expansion level. To maintain the imposed variable ordering the nodes in the function graph have to be exchanged and it is not sufficient to just interpret row as column designators and vice versa. Transposition can be done in $O(|f|)$ time.

Matrix inversion is done by performing Gaussian elimination on the original matrix and the identity matrix at the same time. In other words we solve the system of linear equations $A \star X = 1$ with the use of pivoting and row transformations. The steps required by Gaussian elimination consist of [25]:

- selecting a partial pivot in every step j such that $|a_{pj}| = max_{i \geq j} |a_{ij}|$
- normalizing the selected row by multiplying the row by the inverse of the pivot $\frac{1}{|a_{pj}|}$
- swapping rows j and p according to above pivot selection
- subtracting multiples of the pivot row j from all rows $i > j$ such that $a_{ji} = 0, \forall i > j$

All of the above operations except for row swapping can be implemented efficiently in time $O(|A|)$ or $O(|A| \cdot |R|)$ where R denotes the FEVBDD representing the pivot row. Row swapping is performed by matrix multiplication of matrix A with a permutation matrix P and therefore is of complexity $O(\|A\| \cdot \|P\|)$. Permutation matrices can be obtained by $P_{ij} = I \oplus M_{ij}$ where P_{ij} denotes a permutation matrix swapping rows i and j, I represents the identity matrix and

M_{ij} designates a matrix

$$M = \begin{cases} m_{rs} = 1 & \text{if } r = i, j \text{ or } s = i, j \\ m_{rs} = 0 & \text{otherwise} \end{cases}$$

In general, partial pivoting is done in order to improve the numerical accuracy of Gaussian elimination. Since our implementation relies on fractions of arbitrary precision we always use the exact values and numerical stability is not an issue. In order to avoid unnecessary row swapping we only perform the partial pivoting if it holds in step j that $a_{jj} = 0$.

In addition to the basic matrix operations, fast search operations for specific matrix elements have been implemented. Algorithms for searching both the value and position of the minimal, maximal or absolute maximal element in a given matrix were developed. The algorithm performing the search for the maximal element is shown in tables 4.1 and 4.2. This approach makes use of the *min* and *max* fields that can be associated with every node. The computational complexity for finding both the value and position of the minimal or maximal element in a $n \times n$ matrix is $O(\log_2(\lceil n \rceil))$. We will now explain the basic idea behind the algorithms in the case of searching for the maximal element. Given a FEVBDD node **f** and its two successors $\mathbf{f_t}$ and $\mathbf{f_e}$ we can easily determine which edge leads to the maximal element. Based on the values of the max and min fields of $\mathbf{f_t}$ and $\mathbf{f_e}$ we simply recompute the max field of **f** and select the successor that originally generated the max field of **f**. If only the value of the maximal or minimal element is of interest, it can be computed directly from the min and max field of the top node **f** without any further computation.

```
get_max_element(⟨c_f, w_f, f, rule_f⟩)
{
        *
        * compute the value of the maximal element
        *
        if(w_f < 0) {
                max = c_f + w_f · min_f;
        }
        else {
                max = c_f + w_f · max_f;
        }
        *
        * compute the position of the maximal element
        *
        get_max_position(⟨c_f, w_f, f, rule_f⟩,(row,col));

        return(max,row,col);
}
```

Table 4.1: *get_max_element*

```
get_max_position(⟨c_f, w_f, f, rule_f⟩,(row,col))
{
    if(w_{t_f} < 0) {
        max(t) = ev_f + w_{t_f} · min_{f_t};
        min(t) = ev_f + w_{t_f} · max_{f_t};
    }
    else {
        max(t) = ev_f + w_{t_f} · max_{f_t};
        min(t) = ev_f + w_{t_f} · min_{f_t};
    }
    if(w_{e_f} < 0) {
        max(e) = w_{e_f} · min_{f_e};
        min(e) = w_{e_f} · max_{f_e};
    }
    else {
        max(e) = w_{e_f} · max_{f_e};
        min(e) = w_{e_f} · min_{f_e};
    }
    if(w_f < 0) {
        max(t) = -min_t;
        max(e) = -min_e;
    }
    *
    * select the edge leading to the maximal element
    *
    if(max(t) > max(e) {
        ⟨c_h, w_h, h, rule_h⟩=t_child(⟨c_f, w_f, f, rule_f⟩);
    *
    * update row or column position
    *
        if(IS_ROW(f))
            row=row + submatrix_size / 2;
        else /* IS_COL(f) */
            col=col + submatrix_size / 2;
    }
    else {
        ⟨c_h, w_h, h, rule_h⟩=e_child(⟨c_f, w_f, f, rule_f⟩);
    }
    get_max_position(⟨c_h, w_h, h, rule_h⟩,(row,col));
}
```

Table 4.2: *get_max_position*

value range	EVBDD	FEVBDD	
		GCD	RATIONAL
integer	12 bytes	20 bytes	24 bytes
fractions	16 bytes		24 bytes

Table 4.3: Memory requirement per node

4.3 Experimental Results

We have applied our FEVBDD based matrix package to the problem of solving the Chapman-Kolmogorov equations [22] that arise when computing the global state probabilities of FSMs. Though the memory consumption of our inversion routine is relatively low (8M for inverting a 64x64 matrix), the run time is very high. This is due to several factors. First, the algorithm for Gaussian elimination is purely sequential whereas FEVBDDs are recursively defined. Consequently, computation caching for matrix inversion does not exist. A recursive algorithm for matrix inversion will perform much better on FEVBDDs. Secondly, when using fractions of arbitrary length all arithmetic operations need substantially more time than is necessary for ordinary integers. We therefore use the obtained inverses primarily as examples of real life non-sparse matrices that can be represented compactly using FEVBDDs and compare them with their EVBDD representations. As can be seen from the table below using FEVBDDs gives savings of up to 50% compared to EVBDDs in the number of nodes required to represent the non-sparse inverse. Of course, one has to consider that the storage requirement per node is higher for FEVBDDs than for EVBDDs. An overview of the memory usage per node in the various modes available for EVBDDs and FEVBDDs is given in table 4.3. We assume that every EVBDD node consists of an integer or fractional edge value and two pointers to the children. Every FEVBDD node consists of two fractional edge weights and two pointers in the RATIONAL mode or three integer edge weights and two pointers in the GCD mode.

The total memory consumption for storing the matrices using EVBDDs and FEVBDDs is shown in tables 4.4 and 4.5, respectively. The given memory usage is based on EVBDDs and FEVBDDs using fractions. The FEVBDDs have been generated using the RATIONAL rule. In the case of CK-Equations we have to use fractions for the edge weights since the matrix elements are fractions. As can be seen from the tables, FEVBDDs do better for the inverses but lose for the original matrices in terms of total memory consumption. This is due to the fact that the original matrices are sparse whereas the inverses are non-sparse. In the case of sparse matrices the additional properties of FEVBDDs are not exploited so that EVBDDs and FEVBDDs perform similarly in the number of nodes. FEVBDDs, however, lose in terms of memory requirement because of the higher cost per FEVBDD node. Since EVBDDs do at least as good as MTBDDs this also gives an idea of the performance of FEVBDDs compared to MTBDDs.

CK-Equations	Size	EVBDD				FEVBDD			
		# of Nodes (orig.)	memory (byte)	# of Nodes (inv.)	memory (byte)	# of Nodes (orig.)	memory (byte)	# of Nodes (inv.)	memory (byte)
bbara	10	61	976	92	1472	49	1176	57	1368
bbtas	6	29	464	28	448	20	480	18	432
beecount	7	33	528	43	688	22	528	25	600
cse	16	86	1376	153	2448	63	1512	85	2040
dk14	7	31	496	43	688	24	576	27	648
dk15	4	12	192	12	192	8	192	7	168
dk16	27	150	2400	494	7904	130	3120	314	7536
dk17	8	33	528	41	656	26	624	26	624
dk27	7	24	384	33	528	19	456	24	576
dk512	15	56	896	124	1984	50	1200	85	2040
donfile	24	93	1488	139	2224	81	1944	120	2880
ex1	20	112	1792	279	4464	83	1992	172	4128
ex2	19	113	1808	251	4016	97	2328	156	3744
ex3	10	61	976	119	1904	49	1176	68	1632
ex4	14	48	768	56	896	36	864	42	1008
ex5	9	54	864	93	1488	44	1056	57	1368
ex6	8	40	640	58	928	28	672	31	744
ex7	10	65	1040	105	1680	57	1368	62	1488
keyb	19	133	2128	369	5904	93	2232	212	5088
kirkman	16	25	400	25	400	24	576	22	528
lion	4	12	192	10	160	8	192	7	168
lion9	9	38	608	55	880	31	744	45	1080
mark1	15	57	912	8	128	48	1152	8	192

Table 4.4: Experimental results

CK-Equations	Size	EVBDD				FEVBDD			
		# of Nodes (orig.)	memory (byte)	# of Nodes (inv.)	memory (byte)	# of Nodes (orig.)	memory (byte)	# of Nodes (inv.)	memory (byte)
mc	4	10	160	9	144	8	192	6	144
modulo12	12	31	496	32	512	29	696	27	648
opus	10	56	896	70	1120	47	1128	54	1296
planet	48	209	3344	488	7808	157	3768	366	8784
planet1	48	209	3344	488	7808	157	3768	366	8784
pma	24	132	2112	296	4736	102	2448	182	4368
s1	20	149	2382	352	5632	117	2808	201	4824
s1a	20	149	2382	352	5632	117	2808	201	4824
s1494	48	242	3872	1525	24400	189	4536	841	20184
s208	18	62	992	217	3472	55	1320	98	2354
s27	6	29	464	35	560	22	528	21	504
s386	13	68	1088	146	2336	52	1248	83	1992
s8	5	28	448	28	448	20	480	20	480
sand	32	119	1904	193	3088	86	2064	135	3240
shiftreg	8	26	416	30	480	21	504	22	528
styr	30	170	2720	370	5920	116	2784	226	5424
tav	4	9	144	9	144	8	192	6	144
tbk	32	73	1168	81	1296	62	1488	61	1464
tma	20	114	1824	249	3984	88	2112	155	3720
train11	11	52	832	71	1136	43	1032	57	1368
train4	4	9	144	9	144	8	192	6	144

Table 4.5: Experimental results (cont.)

Chapter 5

Conclusion

We showed that by associating both an additive and a multiplicative weight with the edges of an Edge-Valued Binary Decision Diagram, EVBDDs could successfully be extended to Factored Edge-Valued Binary Decision Diagrams. The new data structure preserves the canonical property of the EVBDD and allows efficient caching of operational results. All properties that have been defined for EVBDDs could be adapted to FEVBDDs. The *additive property* was extended to the *affine property*. It was shown that FEVBDDs provide a more compact representation of arithmetic functions than EVBDDs. Additionally, the complexity of certain operations could be reduced significantly. We showed that FEVBDDs representing boolean functions allow us to incorporate the concept of complement edges that has originally been proposed for OBDDs.

In combination with the FEVBDD package we also implemented an arithmetic package which supplies arithmetic operations on both integers and fractions of arbitrary precision.

A complete matrix package based on FEVBDDs was introduced. We applied the package to solving the Chapman-Kolmogorov equations. The experimental results show that in the majority of cases FEVBDDs win over the corresponding EVBDD representation of the matrices in terms of number of nodes and memory consumption.

Furthermore, we showed that the EVBDD based Integer Linear Programming solver FGILP benefits from using FEVBDDs instead of EVBDDs. FEVBDDs can be used for logic verification but they lose to *BMDs for some word-level operations. Nevertheless, FEVBDDs offer better time complexity for boolean operations compared to *BMDs and improved memory consumption in comparison with EVBDDs.

The primary focus for future research based on this new data structure is to implement the improvements for FGILP and to enhance upon the concept of Structured FEVBDDs, which are based on SEVBDDs. An extension of SFEVBDDs that allows modeling loop statements and recursive functions could be applied to the problem of verifying sequential machines.

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Appendix A

Notation

A.1 OBDD and MTBDD

- The true-edge or 1-edge is denoted by the figure 1 associated with the edge. The else-edge or 0-edge is denoted by the figure 0 associated with the edge.
- $\mathbf{v}$ represents both an OBDD/MTBDD node and the OBDD/MTBDD rooted by node $\mathbf{v}$
- *index(*$\mathbf{v}$*)* : index of the variable associated with node $\mathbf{v}$. If $\mathbf{v}$ is a terminal node, then $index(\mathbf{v}) = n$.
- $$t\_child(\mathbf{v}, i) = \begin{cases} child_t(\mathbf{v}) & \text{if } index(\mathbf{v}) = i, \\ \mathbf{v} & \text{otherwise.} \end{cases}$$
  $$e\_child(\mathbf{v}, i) = \begin{cases} child_e(\mathbf{v}) & \text{if } index(\mathbf{v}) = i, \\ \mathbf{v} & \text{otherwise.} \end{cases}$$
- $new\_obdd(x, \mathbf{t}, \mathbf{e})$ returns an OBDD node $\mathbf{v}$ such that $variable(\mathbf{v}) = x$, $child_t(\mathbf{v}) = \mathbf{t}$, and $child_e(\mathbf{v}) = \mathbf{e}$.

A.2 EVBDD

- The true-edge is denoted by an edge value associated with the edge. The else-edge is denoted by a plain edge.

- The pair $\langle c_f, \mathbf{f}\rangle$ represents the EVBDD. c_f denotes the constant value and $\mathbf{f}$ stands for the DAG. Usually c_f is depicted as a dangling incoming edge to the root of the DAG.

- *index(*$\mathbf{v}$*)* : index of the variable associated with node $\mathbf{v}$. If $\mathbf{v}$ is a terminal node, then $index(\mathbf{v}) = n$.

-

$$t\_child(\langle c_f, \mathbf{f}\rangle, i) = \begin{cases} \langle c_f + ev, child_t(\mathbf{f})\rangle & \text{if } index(\mathbf{v}) = i, \\ \langle c_f, \mathbf{f}\rangle & \text{otherwise.} \end{cases}$$
$$e\_child(\langle c_f, \mathbf{f}\rangle, i) = \begin{cases} \langle c_f, child_e(\mathbf{f})\rangle & \text{if } index(\mathbf{f}) = i, \\ \langle c_f, \mathbf{f}\rangle & \text{otherwise.} \end{cases}$$

- *new_evbdd*(x,$\langle c_t, \mathbf{t}\rangle$,$\langle c_e, \mathbf{e}\rangle$) returns an EVBDD node $\langle c_f, \mathbf{f}\rangle$ such that $variable(\mathbf{f}) = x$, $c_f = c_e$, $ev = c_t - c_e$, $child_t(\mathbf{f}) = \mathbf{t}$, and $child_e(\mathbf{f}) = \mathbf{e}$.

A.3 *BMD

- For a given node $\mathbf{v}$ in the *BMD the left edge always leads to the *linear moment* while the right edge leads to the *constant moment*. A multiplicative weight may be associated with the linear-edges.[1]

- The tuple $\langle w_v, \mathbf{v}, rule_v\rangle$ represents the *BMD. w_v is the multiplicative weight. $\mathbf{v}$ represents the DAG and $rule_f$ gives the employed weight normalizing rule. w_f is depicted as a dangling incoming edge to the root of the DAG.

- *index(*$\mathbf{v}$*)* : index of the variable associated with node $\mathbf{v}$. If $\mathbf{v}$ is a terminal node, then $index(\mathbf{v}) = n$.

-

$$l\_child(\langle w_v, \mathbf{v}, rule_v\rangle, i) = \begin{cases} \langle w_v \cdot w_{l_v}, child_l(\mathbf{v}), rule\rangle & \text{if } index(\mathbf{v}) = i, \\ \langle w_v, \mathbf{v}, rule_v\rangle & \text{otherwise.} \end{cases}$$
$$c\_child(\langle w_v, \mathbf{v}, rule_v\rangle, i) = \begin{cases} \langle w_v, child_c(\mathbf{v}), rule\rangle^2 & \text{if } index(\mathbf{v}) = i, \\ \langle w_v, \mathbf{v}, rule_v\rangle & \text{otherwise.} \end{cases}$$

[1]In integer mode a multiplicative weight may also be associated with the constant-edges.

- *new_*bmd*(x,$\langle w_l, \mathbf{l}, rule_l\rangle$,$\langle w_c, \mathbf{c}, rule_c\rangle$) returns a *BMD node $\langle w_v, \mathbf{v}, rule_v\rangle$ such that $variable(\mathbf{v}) = x$, $w_v = norm\_weight(w_l, w_c)$, $w_l \leftarrow \frac{w_l}{w_v}$, $w_c \leftarrow \frac{w_c}{w_v}$, $child_l(\mathbf{v}) = \mathbf{l}$, and $child_c(\mathbf{v}) = \mathbf{c}$.

A.4 FEVBDD

- The true-edge is denoted by a pair of edge values (ev, w) associated with the edge where ev denotes the additive weight and w denotes the multiplicative weight. The else-edge is denoted by a plain edge.[3]

- The tuple $\langle c_f, w_f, \mathbf{f}, rule_f\rangle$ represents the FEVBDD. c_f is the additive weight and w_f is the multiplicative weight. $\mathbf{f}$ represents the DAG and $rule_f$ gives the employed weight normalizing rule. The pair (c_f, w_f) is depicted as a dangling incoming edge to the root of the DAG.

- *index(*$\mathbf{v}$*)* : index of the variable associated with node $\mathbf{v}$. If $\mathbf{v}$ is a terminal node, then $index(\mathbf{v}) = n$.

-
$$
\begin{aligned}
t\_child(\langle c_f, w_f, \mathbf{f}, rule_f\rangle, i) &= \begin{cases} \langle c_f + w_f \cdot ev, w_f \cdot w_{t_f}, child_t(\mathbf{f}), rule\rangle & \text{if } index(\mathbf{f}) = i, \\ \langle c_f, w_f, \mathbf{f}, rule_f\rangle & \text{otherwise.} \end{cases} \\
e\_child(\langle c_f, w_f, \mathbf{f}, rule_f\rangle, i) &= \begin{cases} \langle c_f, w_f, child_e(\mathbf{f}), rule\rangle^4 & \text{if } index(\mathbf{f}) = i, \\ \langle c_f, w_f, \mathbf{f}, rule_f\rangle & \text{otherwise.} \end{cases}
\end{aligned}
$$

- *new_fevbdd*(x,$\langle c_t, w_t, \mathbf{t}, rule_t\rangle$,$\langle c_e, w_e, \mathbf{e}, rule_e\rangle$) returns a FEVBDD node $\langle c_f, w_f, \mathbf{f}, rule_f\rangle$ such that $variable(\mathbf{f}) = x$, $c_f = c_e$, $ev = \frac{c_t - c_e}{w_f}$, $w_f = norm\_weight(c_t - c_e, w_t, w_e)$, $w_t \leftarrow \frac{w_t}{w_f}$, $w_e \leftarrow \frac{w_e}{w_f}$, $child_t(\mathbf{f}) = \mathbf{t}$, and $child_e(\mathbf{f}) = \mathbf{e}$.

---

[2] In case of the integer mode: $\langle w_v \cdot w_{c_v}, child_c(\mathbf{v}), rule\rangle$.

[3] In case of the GCD mode the else-edge has a single multiplicative weight associated with it.

[4] In case of the GCD mode: $\langle c_f, w_f \cdot w_{e_f}, child_e(\mathbf{f}), rule\rangle$.

Appendix B

Complexity Analysis

B.1 OBDD and MTBDD

- Cardinality of a node $\mathbf{v}$ with $variable(\mathbf{v}) = x_i$ in $\mathbf{f}$

$$\|\mathbf{v}\| = |\mathbf{v}| = 1$$

- Cardinality of $\mathbf{f}$

$$\|\mathbf{f}\| = |\mathbf{f}| = \sum_{\mathbf{v} \in \mathbf{f}} \tag{B.1}$$

B.2 EVBDD

- Domain of a node $\mathbf{v}$ with $variable(\mathbf{v}) = x_i$ in $\langle c_f, \mathbf{f} \rangle$

$$D_{\mathbf{v}}^{EVBDDeval} = \{c_u | \langle c_u, \mathbf{u} \rangle = EVBDDeval(\langle c_f, \mathbf{f} \rangle, \Phi) where\ \mathbf{u} = \mathbf{v}, \forall \Phi \in \{0,1\}^i\} \tag{B.2}$$

- Cardinality of a node $\mathbf{v}$ with $variable(\mathbf{v}) = x_i$ in $\langle c_f, \mathbf{f} \rangle$

$$\|\mathbf{v}\| = |D_{\mathbf{v}}^{EVBDDeval}|$$

- Cardinality of $\langle c_f, \mathbf{f} \rangle$

$$\|\langle c_f, \mathbf{f} \rangle\| = \sum_{\mathbf{v} \in \mathbf{f}} \|\mathbf{v}\| \tag{B.3}$$

B.3 FEVBDD

- Domain of a node $\mathbf{v}$ with $variable(\mathbf{v}) = x_i$ in $\langle c_f, w_f, \mathbf{f}, rule_f \rangle$

$$\begin{aligned} D_{\mathbf{v}}^{FEVBDDeval} &= \\ &\{(c_u, w_u) | \langle c_u, w_u, \mathbf{u}, rule_u \rangle = FEVBDDeval(\langle c_f, w_f, \mathbf{f}, rule_f \rangle, \Phi) \\ &where\ \mathbf{u} = \mathbf{v}, \forall \Phi \in \{0,1\}^i\} \end{aligned} \tag{B.4}$$

- Cardinality of a node $\mathbf{v}$ with $variable(\mathbf{v}) = x_i$ in $\langle c_f, w_f, \mathbf{f}, rule_f \rangle$

$$\|\mathbf{v}\| = |D_{\mathbf{v}}^{FEVBDDeval}|$$

- Cardinality of $\langle c_f, w_f, \mathbf{f}, rule_f \rangle$

$$\|\langle c_f, w_f, \mathbf{f}, rule_f \rangle\| = \sum_{\mathbf{v} \in \mathbf{f}} \|\mathbf{v}\| \tag{B.5}$$

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