Performance of a Propagation Delay Tolerant ALOHA Protocol for Underwater Wireless Networks

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Abstract

We analyze a propagation delay tolerant ALOHA (PDT-ALOHA) protocol proposed recently for underwater networks [16]. In this scheme, guard-bands are introduced at each slot to reduce collisions between senders with different distances to the receiver. We prove some interesting properties concerning the performance of this protocol and show how it varies with key application and protocol parameters such as maximum propagation delay, traffic load, and the guard-band. Although it turns out that exact expressions for the maximum throughput of PDT-ALOHA can be quite complicated, we propose a useful simple expression which is shown numerically to be a very good approximation.

I. Introduction

Underwater wireless networks have recently gained a great deal of attention as a topic of research, with a wide range of sensing and security applications just starting to be explored [3], [11], [10]. Since radio-frequency (RF) electromagnetic waves, except those of very low frequencies ($30 \sim 300$ Hz), decay very rapidly in water [4], underwater networks need to adopt acoustic transmissions instead in most cases. However, the acoustic transmission introduces many kinds of problems so that a lot of high technologies developed for the RF transmission are stumbled in this environment.

The low bandwidth of the acoustic transmission dramatically depends on both transmission range and frequency [7]; most acoustic tranceivers operate below 30kHz [7] and nearly no research nor commercial system can exceed $40km * kb/s$ as the maximum attainable *range*rate* product [4]. Due to this low bandwidth the frequency division multiple access (FDMA) scheme is not suitable [7]. The acoustic transmission also has the large propagation delay which makes contention-based protocols relying on carrier sensing and handshaking unappropriate [12], [2]. It also hampers the use of time division multiple access (TDMA) scheme because TDMA then requires the long time guards [7], [4]. Therefore, the design and analysis of new approaches are required for different layers, including medium access.

The problem of designing a simple medium access protocol appropriate for underwater networks was addressed recently in [16]. The authors of this work show that the performance of classical slotted ALOHA deteriorates in an underwater setting where transmissions from one slot can overlap with future slots. They propose the introduction of guard-bands in each slot to address this problem; we refer to their scheme as the propagation delay tolerant ALOHA (PDT-ALOHA) protocol.

In PDT-ALOHA, the guard-band in each slot is designed in order to relieve collisions between packets of different senders transmitted in consecutive time slots. Increasing the guard-band reduces collisions, but it also increases the length of each time slot, potentially reducing the throughput. Thus the selection of the appropriate guard-band length to get the maximum throughput can be formulated as an optimization problem.

We analyze mathematically the throughput of the PDT-ALOHA protocol given a guard band, a traffic load, and the maximum propagation delay. Then, we investigate its maximum throughput over the guard bands and the traffic loads given the maximum propagation delay. Although it turns out to be difficult to derive an exact expression for the maximum throughput, we propose simple approximate expressions which are very close to the maximum throughput calculated through numerical methods. We also prove a number of interesting and useful properties concerning the performance of the PDT-ALOHA protocol.

Related Works

There are many literatures which have studied on the throughput of ALOHA protocols in the underwater networks. [9] has investigated the impact of the large propagation delay on the throughput of selected classical MAC protocols and their variants through simulations. [18] has compared the performance of ALOHA and CSMA with RTS/CTS protocols in underwater wireless networks. And [16] has studied on the throughput of PDT-ALOHA through simulations producing rough idea of the performance. While these works are mainly based on simulations we approach the problem from the theoretical view point.

There have been other works which take the theoretical approach as we do. [17] has analyzed slotted ALOHA without the guard band and concluded that slotted ALOHA degrades to unslotted ALOHA under high propagation delay. [8] have analyzed the performance of ALOHA in a linear multi-hop topology. However, these works do not consider the guard band to relieve the negative effect of the large propagation delay. [5], [6] have taken consideration of the guard band for the slotted ALOHA protocols in their analysis. However, they have assumed satellite networks where the imperfection or sloppiness of each node's implementation causes variable propagation delay, and they have focused on how to deal with the sloppiness using the guard bands. The main difference from their problem is that nodes are located on the ground approximately same distance away from the satellite in their problem so that the propagation delay is more or less same for each node. But, the distance to the receiver can vary greatly from node to node in underwater wireless sensor networks.

II. Assumptions

We made following assumptions to analyze the performance of the PDT-ALOHA protocol.

- The network has one receiver and n transmitters, which are deployed in the two-dimensional disk area.
- The receiver locates at the center of the disk area, and the transmitters are deployed uniformly at random in the area.
- The propagation speed of communication is positive finite constant regardless of the location in the network, so that the maximum propagation time from the receiver to the farthest transmitter is a positive finite constant τ_m.
- The transmission rate is constant for every transmitter.
- The packet size is constant so that, along with the constant transmission rate, the transmission time for a packet is constant T.
- The traffic in the network is I.I.D. Bernoulli so that a transmitter sends a packet to the receiver with probability p in each time slot.
- If the receiver receives more than one packet simultaneously at any time in a time slot, all the packets involved fail to get delivered successfully causing a collision.
- The links over which transmission takes place are lossless (e.g., using blacklisting).
- A transmitter always transmits a packet at the start of the time slot if the transmitter wants to send the packet.
- All the nodes have the globally synchronized time slots.
- The transmission time T is no less than the maximum propagation time τ_m.

The assumption that $T \geq \tau_m$ is to make sure that the collision between a time slot and another is confined to the consecutive time slots. So, with this assumption, there is no possibility that a packet sent in i-th time slot collides with another in j-th time slot, where $j \notin \{i-1, i, i+1\}$. In addition, this assumption is quite reasonable in real system because a system that has the propagation time larger than the transmission time will suffer from a poor throughput.

III. Throughput Analysis

In this section we analyze the throughput of the PDT-ALOHA protocol. Let us first look at the time slot. Each time slot consists of a transmission time and a guard band following the former. Since the guard band of the size of maximum propagation time τ_m would eliminate all the collision between different time slots, it does not make sense to have the guard band whose size is more than τ_m only decreasing the throughput without any gain. Hence, we use the normalized factor $\beta\ s.t.\ 0 \leq \beta \leq 1$ in expressing the size of guard band so that the time slot size is $T + \beta \cdot \tau_m$.

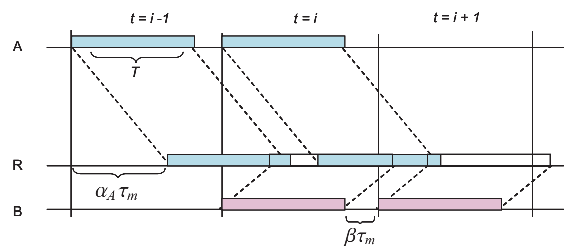


Fig. 1. Time diagram of packet transmission; A and B are transmitters and R is the receiver. B locates closer to the receiver than A.

A. Expected Number of Successful Packet Receptions

In order to analyze the throughput we first derive the expected number of successful packet receptions in a time slot. We use the linearity of expectations and conditional probabilities to calculate the expected number. Let the indicator variable I_i denote whether or not the receiver receives the packet from i-th transmitter successfully in the time slot.

$$I_i = \begin{cases} 1, & \textit{if successful reception} \\ 0, & \textit{otherwise} \end{cases} \tag{1}$$

Let N denote the random variable of the number of successful reception. Then, $N = \sum_i I_i$. Hence, the expected number is, by the linearity of expectations and conditional probability, as follows;

$$\begin{aligned} E[N] &= \sum_{i=1}^{n} E[I_i] = \sum_{i=1}^{n} \Pr\{I_i = 1\} \\ &= \sum_{i=1}^{n} \Pr\{\textit{no collision} \mid \textit{i-th sender sends}\} \cdot \Pr\{\textit{i-th sender sends}\} \\ &= n \cdot p \cdot \Pr\{NC|n_i\} \end{aligned} \tag{2}$$

where $NC|n_i$ denotes the event that no collision occurs given that i-th sender transmits. The last equality of the above equations holds since the collision probability is symmetrical among all the senders.

Therefore, in order to calculate the expected number, we only need to find out the probability of no collision for the transmitted packet from the i-th sender whose location is uniform at random over the network area.

B. Probability of No Collision

Let us consider some situations where collision can occur in order to get some intuition. Suppose a simple network with two senders, A and B, and one receiver R. A locates right next to R while B is very far from R, and the size of guard band is small enough. Then, if B transmits in the i-th time slot, R would receive last part of the packet in the beginning of the $(i+1)$-th slot, which would produce collision with the packet transmitted in the $(i+1)$-th slot by A although the two packets are sent in different time slots. The time diagram in Figure 2 visually shows this situation, where α_i is the normalized propagation time distance of $i \in \{A, B\}$ from R defined by Definition 1, the normalized guard band size β is less than 1, and $\alpha_A > \alpha_B + \beta$. However, if $\alpha_A = \alpha_B$ there is no collision between packets in different time slots. Therefore, we can see that the collision depends on nodes' locations and two packets transmitted in different time slots can experience collision between each other.

Definition 1: The **normalized (propagation) time distance** of sender X from the receiver is the propagation time from the receiver to X divided by the maximum propagation time τ_m in the network.

After all, it is not hard to identify collision regions $R_p(\alpha), R_c(\alpha), R_n(\alpha)$ for the interested transmitter I which has the normalized time distance of α, where $R_p(\alpha)$ denotes the region such that a packet sent from I collides with a packet sent in the previous consecutive time slot by a node in $R_p(\alpha)$; $R_c(\alpha)$ in the same time slot; and

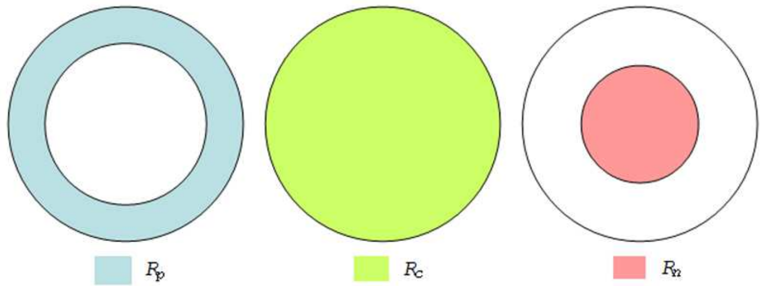


Fig. 2. Collision regions

$R_n(\alpha)$ in the next consecutive time slot. Equation (3), (4), (5) specify the regions in terms of the normalized time distance and Figure 2 visually presents the regions.

$$R_p(\alpha) = \{\tau_p|\alpha+\beta \leq \tau_p \leq 1\} \quad (3)$$
$$R_c(\alpha) = \{\tau_c|0 \leq \tau_c \leq 1\} \quad (4)$$
$$R_n(\alpha) = \{\tau_n|0 \leq \tau_n \leq \alpha-\beta\} \quad (5)$$

The probability of no collision given a packet sent by an arbitrary i-th sender n_i is then as follows conditioning on the n_i's normalized time distance α;

$$\begin{aligned}\Pr\{NC|n_i\} &= \int_0^1 \Pr\{NC|n_i, n_i \text{ at } \alpha \text{ away}\} \cdot \textit{pdf}\{\alpha \text{ away} \mid n_i \text{ trans.}\} d\alpha \\ &= \int_0^1 2\alpha \Pr\{NC|\alpha\} d\alpha \quad (6)\end{aligned}$$

The last equation holds because the location of a node is independent of the packet transmission in our assumption.

Meanwhile, the probability of (no) collision of a specific packet does depend on the location of its sender because it defines the three collision regions, $R_n(\alpha), R_c(\alpha),$ and $R_p(\alpha)$, and the regions' areas affect the probability. Hence, further conditioning on the numbers of transmitters in those three collision regions, we can get the following equation:

$$\begin{aligned}\Pr\{NC|n_i, n_i \text{ at } \alpha \text{ away}\} &= \Pr\{NC| \text{ sent at } \alpha \text{ away}\} \\ &= \sum_{\substack{0 \leq x \leq n-1, \\ 0 \leq y \leq n-1, \\ 0 \leq z \leq n-1}} \begin{matrix}\Pr\{NC \mid \alpha, N_n = x, N_c = y, N_p = z\} \\ \times P\{N_n = x, N_c = y, N_p = z|\alpha\}\end{matrix} \\ & \quad (\text{s.t } z = n-1-x-y) \quad (7)\end{aligned}$$

where $N_n, N_c,$ and N_p denote the number of transmitter in $R_n, R_c,$ and R_p respectively. Note that there are $(n-1)$ other transmitters (or interferers) because we focus on one specific transmitter's success.

Note also that the event of $N_n = x, N_c = y, N_p = z|\alpha$ has the multinomial distribution with parameters $n-1$, $p_n(\alpha)$, $p_c(\alpha)$, and $p_p(\alpha)$, where $p_i(\alpha),\ i \in \{n, c, p\}$ denotes the probability that a transmitter lies in $R_i(\alpha)$. And each of these probabilities is the ratio of its area to the entire area of the network;

$$p_n(\alpha) = \begin{cases}(\alpha-\beta)^2 & , \textit{if } 0 < \alpha-\beta < 1 \\ 0 & , \textit{otherwise}\end{cases}$$
$$p_c(\alpha) = 1$$
$$p_p(\alpha) = \begin{cases}1-(\alpha+\beta)^2 & , \textit{if } 0 < \alpha+\beta < 1 \\ 0 & , \textit{otherwise}\end{cases}$$

Now we have two cases, each of which has three sub-cases. In the first case A ($0 \leq \beta \leq 0.5$), we have three sub-cases; (i) $0 \leq \alpha \leq \beta$, where $R_n(\alpha) = \emptyset$ so that $p_n(\alpha) = 0$; (ii) $\beta \leq \alpha \leq 1-\beta$, where all three collision

$$\Pr\{NC|n_i\} = \frac{(1-p)^{n-1}}{8np^2}\left(-8p\left(-2np\,{}_2F_1\left(\frac{1}{2},1-n;\frac{3}{2};\frac{p\beta^2}{p-1}\right)\beta^2+4np\,{}_2F_1\left(\frac{1}{2},1-n;\frac{3}{2};\frac{4p\beta^2}{p-1}\right)\beta^2\right.\right.$$
$$\left.+(p-1)\left(\left(\frac{-4p\beta^2+p-1}{p-1}\right)^n-\left(\frac{-p\beta^2+p-1}{p-1}\right)^n\right)\right)(1-p)^{n-1}$$
$$+\frac{(1-2\beta)\left((p(4n(\beta-1)\beta-1)+1)\left(1-p(1-2\beta)^2\right)^n+\left(4np\beta^2+p-1\right)\left(p\left(4\beta^2-1\right)+1\right)^n\right)}{(n+1)\beta^2(2\beta-1)}$$
$$+8p\left(\left(1-p(1-2\beta)^2\right)^n-\left(1-p(\beta-1)^2\right)^n+2np\beta(2\beta-1)\,{}_2F_1\left(\frac{1}{2},1-n;\frac{3}{2};p(1-2\beta)^2\right)\right.$$
$$\left.\left.-2np(\beta-1)\beta\,{}_2F_1\left(\frac{1}{2},1-n;\frac{3}{2};p(\beta-1)^2\right)\right)\right) \tag{11}$$

Case A: when $0 \leq \beta \leq 0.5$

$$\Pr\{NC|n_i\} = 2(1-p)^{n-1}\left(\beta-\frac{\left(1-p(\beta-1)^2\right)^n+2np(\beta-1)\beta\,{}_2F_1\left(\frac{1}{2},1-n;\frac{3}{2};p(\beta-1)^2\right)-1}{2np}\right.$$
$$-\beta(1-p)^{n-2}\,{}_2F_1\left(\frac{1}{2},1-n;\frac{3}{2};\frac{p}{p-1}\right)+\beta^2(1-p)^{n-1}\,{}_2F_1\left(\frac{1}{2},1-n;\frac{3}{2};\frac{p\beta^2}{p-1}\right)$$
$$\left.-\frac{1}{2np}\left(\left(p\left(\beta^2-1\right)+1\right)^n-1\right)-\frac{1}{2}\right) \tag{12}$$

Case B: when $0.5 < \beta \leq 1$

regions can have areas larger than zero; and (iii) $1-\beta \leq \alpha \leq 1$, where $R_p(\alpha)=\emptyset$ so that $p_p(\alpha)=0$. In the other case B ($0.5 \leq \beta \leq 1$), we have another three sub-cases; (i) $0 \leq \alpha \leq 1-\beta$, where $R_n(\alpha)=\emptyset$; (ii) $1-\beta \leq \alpha \leq \beta$, where $R_p(\alpha)=R_n(\alpha)=\emptyset$; and (iii) $\beta \leq \alpha \leq 1$, where $R_p(\alpha)=\emptyset$.

Let us consider Case A.(i) first. In this case, the conditional probability of no collision turns out to involve the binomial series as follows;

$$\begin{aligned}\Pr\{NC|\alpha\} &= \sum_{z=0}^{n-1}\Pr\{NC|\alpha,N_p=z,N_c=n-1,N_n=0\}\cdot\Pr\{N_p=z|\alpha\} \\ &= \sum_{z=0}^{n-1}\binom{n-1}{z}(1-p)^z(1-p)^{n-1}\cdot(1-(\alpha+\beta)^2)^z((\alpha+\beta)^2)^{n-1-z} \\ &= (1-p)^{n-1}(1-p+p(\alpha+\beta)^2)^{n-1}\end{aligned} \tag{8}$$

Equation (9) and (10) are the summary after calculating the other cases in the similar way; most of them involve the binomial series although Case A.(ii) involves the multinomial series.

In Case A,

$$\Pr\{NC|\alpha\}=\begin{cases}(1-p)^{n-1}(1-p+p(\alpha+\beta)^2)^{n-1}, & \textit{if } 0\leq\alpha\leq\beta\\ (1-p)^{n-1}(1-p+4p\alpha\beta)^{n-1}, & \textit{if } \beta\leq\alpha\leq 1-\beta\\ (1-p)^{n-1}(1-p(\alpha-\beta)^2)^{n-1}, & \textit{if } 1-\beta\leq\alpha\leq 1\end{cases} \tag{9}$$

Along with Equation (6), the probability of no collision for a specified transmitter is given in Equation (11), where ${}_2F_1(a,b;c;d)$ is the Hypergeometric function.

In Case B,

$$\Pr\{NC|\alpha\}=\begin{cases}(1-p)^{n-1}(1-p+p(\alpha+\beta)^2)^{n-1}, & \textit{if } 0\leq\alpha\leq 1-\beta\\ (1-p)^{n-1}, & \textit{if } 1-\beta\leq\alpha\leq\beta\\ (1-p)^{n-1}(1-p(\alpha-\beta)^2)^{n-1}, & \textit{if } \beta\leq\alpha\leq 1\end{cases} \tag{10}$$

And the probability of no collision in this case is given in Equation (12). Although the equations (11) and (12) are quite complicated, they are closed form expression in terms of the Hypergeometric function and fairly fast to evaluate using the numerical method.

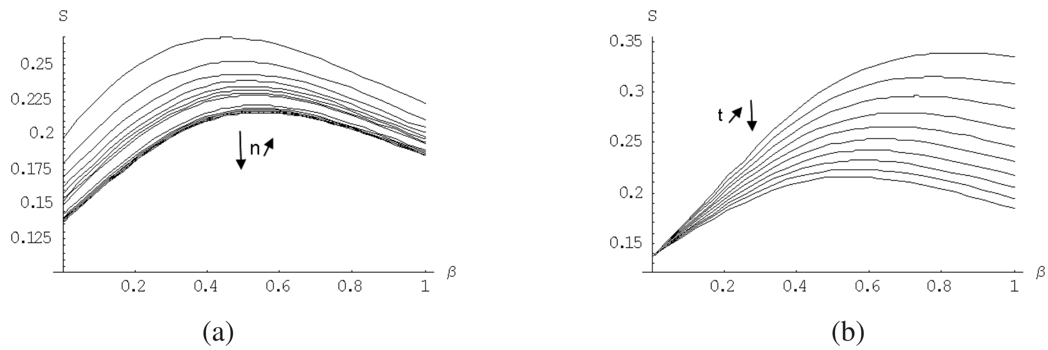


Fig. 3. Throughput of PDT-ALOHA vs. β; (a) when $t = 1$ and n is variable, (b) when $n = 100$ and t is variable.

Note that these expressions of probability of no collision do not involve the maximum propagation delay τ_m implying the probability is independent of τ_m so that the expected number of successful reception is also independent of τ_m. It turns out from Theorem 1 that the expected number is independent of τ_m even after relaxing the assumption of 2D unit disk of the network and the identical distribution of packet transmission for each node.

Theorem 1: Given a network of nodes with fixed spatial locations of nodes, a fixed transmission probability p_i in a time slot for each node i, and a transmission time T for a packet, the expected number of successful packet reception f in a time slot is independent of the maximum propagation time τ_m in the network as long as $0 < \tau_m \leq T$. In other words, it is independent of the propagation speed v_p.

Proof: Since the spatial locations of nodes are fixed, the spatial distance r_m from the receiver to the farthest node is constant;

$$r_m = \tau_m \cdot v_p = \textit{const.}$$

The spatial distance r_i of an arbitrary i-th transmitter is also fixed, and so the normalized propagation time delay α_i of the node is constant regardless of r_m as long as $r_m > 0$ or $0 < v_p < \infty$ because of the following:

$$r_i = \alpha_i \cdot \tau_m \cdot v_p = \alpha_i \cdot r_m \quad \Rightarrow \quad \alpha_i = \frac{r_i}{r_m} = \textit{const.}$$

Let $r(R_i)$ denote the spatial region associate with the collision region R_i. Then, the spatial region of R_n, R_c, and R_p are all fixed regardless of τ_m because

$$\begin{aligned} r(R_n) &= \{r : 0 \leq r \leq (\alpha_i - \beta)\tau_m v_p = (\alpha_i - \beta) r_m\} \\ r(R_c) &= \{r : 0 \leq r \leq \tau_m v_p = r_m\} \\ r(R_p) &= \{r : (\alpha_i + \beta) r_m \leq r \leq r_m\} \end{aligned}$$

and α_i, β, and r_m are all constants.

Hence, the number of nodes in each of R_n, R_c, and R_p is constant regardless of the speed of propagation, and so the probability of no collision of the i-th transmitter is constant. Therefore,

$$f = \sum_i p_i \Pr\{NC|n_i\} = \textit{const. with respect to } \tau_m$$

■

C. Throughput for finite number of nodes

In this paper we consider the throughput S in packets per packet length. Because the expected number of successful packet receptions $f(n, \beta, p)$ in a time slot is independent of the propagation time as long as it is positive finite (Theorem 1), S can be expressed as follows after introducing a new variable $t = \tau_m/T$ the ratio of the maximum propagation delay to the transmission time of a packet:

$$S(n, \beta, p, t) = \frac{f(n, \beta, p)}{1 + \beta t} = \frac{np \Pr\{NC|n_i\}}{1 + \beta t} \tag{13}$$

$$\Pr\{NC|n_i\} = \frac{1}{8\beta^2\lambda^2}e^{-2\left(2\lambda\beta^2+\lambda\right)}\left(-8e^{4\beta^2\lambda}\sqrt{\pi}\lambda^{3/2}\left(e^{\lambda}\left(\text{erf}\left((1-2\beta)\sqrt{\lambda}\right)+\text{erf}\left((\beta-1)\sqrt{\lambda}\right)\right)\right.\right.$$
$$\left.-\text{erfi}\left(\beta\sqrt{\lambda}\right)+\text{erfi}\left(2\beta\sqrt{\lambda}\right)\right)\beta^3-8e^{5\beta^2\lambda}\lambda\beta^2-8e^{\beta(3\beta+2)\lambda}\lambda\beta^2+e^{8\beta^2\lambda}\left(4\lambda\beta^2+1\right)$$
$$\left.+e^{4\beta\lambda}(4\beta(\beta+1)\lambda-1)\right) \tag{17}$$

Case A: when $0 \leq \beta \leq 0.5$

$$\Pr\{NC|n_i\} = \frac{1}{\lambda}e^{(-(\beta-2)\beta-2)\lambda}\left(e^{(\beta-2)\beta\lambda}\left(e^{\lambda}((2\beta-1)\lambda+2)-\sqrt{\pi}\beta\sqrt{\lambda}\left(e^{\lambda}\text{erf}\left((\beta-1)\sqrt{\lambda}\right)\right.\right.\right.$$
$$\left.\left.\left.+\text{erfi}\left(\sqrt{\lambda}\right)-\text{erfi}\left(\beta\sqrt{\lambda}\right)\right)\right)-e^{2(\beta-1)\beta\lambda}-1\right) \tag{18}$$

Case B: when $0.5 < \beta \leq 1$

where we know the probability of no collision from the previous section.

Using the numerical evaluation of Equation (13), Figure 3 shows the characteristics of the throughput depending on the size of guard band β; in (a) the relative maximum propagation delay t is fixed, but the number of nodes n is varying from 3 to 10 with the interval of 1 and then, from 20 to 100 with the interval of 10. In (b) n is fixed but t is varying from 0.1 to 1 with the interval of 0.1. These plots show how the throughput responds to the variables; the optimizer β values are similar for one case, but different in the other case.

D. Throughput of Infinite Number of Nodes

In this section, we investigate the throughput of PDT-ALOHA protocol with the infinite number of nodes with the traffic load λ over the network, i.e, $n \to \infty$ while $p = \lambda/n$. Hence, the throughput in this case is given by

$$S' = \lim_{n\to\infty} S|_{p=\frac{\lambda}{n}} = \frac{\lambda}{1+\beta t}\lim_{n\to\infty}\Pr\{NC|n_i\} \tag{14}$$

Because the integrand of Equation (6) converges uniformly over $[0,1]$ (see Appendix A), we can exchange integral and limitation by Theorem 7.16 of [14]. Hence, with the equalities in Table I, we can achieve the conditional probability of no collision in this limiting case as follows:

If $0 \leq \beta \leq 0.5$;

$$\Pr\{NC|\alpha\} = \begin{cases} e^{-2\lambda+\lambda(\alpha+\beta)^2}, & 0 \leq \alpha \leq \beta \\ e^{-2\lambda+4\lambda\alpha\beta}, & \beta \leq \alpha \leq 1-\beta \\ e^{-\lambda-\lambda(\alpha-\beta)^2}, & 1-\beta \leq \alpha \leq 1 \end{cases} \tag{15}$$

If $0.5 < \beta \leq 1$;

$$\Pr\{NC|\alpha\} = \begin{cases} e^{-2\lambda+\lambda(\alpha+\beta)^2}, & 0 \leq \alpha \leq 1-\beta \\ e^{-\lambda}, & 1-\beta \leq \alpha \leq \beta \\ e^{-\lambda-\lambda(\alpha-\beta)^2}, & \beta \leq \alpha \leq 1 \end{cases} \tag{16}$$

Therefore, the probability of no collision for a packet of a transmitter is given in Equation (17) and (18) for the case A ($0 \leq \beta \leq 0.5$) and the case B ($0.5 < \beta \leq 1$), respectively; erf(z) is the error function and erfi(z) is the imaginary error function [1]. Although the exact solution is quite complicated, calculating it is reasonably fast with numerical methods. And we shall use this to investigate the properties of maximum throughput and its approximation which has a very simple expression (section IV-B).

$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n-1}$	$= e^{-\lambda}$
$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}+\frac{\lambda}{n}(\alpha+\beta)^2\right)^{n-1}$	$= e^{-\lambda+\lambda(\alpha+\beta)^2}$
$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}+4\frac{\lambda}{n}\alpha\beta\right)^{n-1}$	$= e^{-\lambda+4\lambda\alpha\beta}$
$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}(\alpha-\beta)^2\right)^{n-1}$	$= e^{-\lambda(\alpha-\beta)^2}$

TABLE I
EQUALITIES TO USE AS BUILDING BLOCKS TO CALCULATE THE THROUGHPUT IN THE LIMITING CASE

IV. OPTIMIZATION

In this section we investigate the *maximum* number of successful packet receptions in a time slot and the *maximum* throughput of PDT-ALOHA protocol. Although we assume in this section the limiting case where the number of nodes in the network is infinite, it is fairly straightforward to adapt the method we used here for the finite number of nodes. Because the objective functions are very complicated (see Equation (17) and (18)) we resort to use the numerical method and achieve approximations except special cases; $\beta = 0$, or $\beta = 1$, which is dealt with in section IV-C.

A. Maximum Expected Number of Successful Packet Receptions

First we present the analytic findings about the properties of the maximum expected number of successful receptions. The findings are more general than what we assume throughout this paper. We prove through Theorem 2 that the maximum expected number of receptions is monotonically non-decreasing with respect to the guard band β even when the network area is no longer 2D disk and the sending probability is not identical for each node as long as the maximum propagation delay is less than the transmission time of a packet.

Lemma 1: Given the arbitrary location distribution of n transmitters and the probability p_i that i-th transmitter transmits a packet in a time slot, the expected number of successful reception in a time slot, $f(\beta, \vec{p})$, is monotonically non-decreasing as the normalized guard band β increases when the maximum propagation delay τ_m in the network is such that $\tau_m < T$.

In other words,

$$0 \le \beta_1 \le \beta_2 \le 1 \quad \Rightarrow \quad f(\beta_1, \vec{p}) \le f(\beta_2, \vec{p}),$$
$$\textit{for all } \vec{p} = (p_1, \ldots, p_n) \textit{ s.t } 0 \le p_i \le 1,\ \forall i \in \{1, \ldots, n\}$$

Proof: Because $\tau_m < T$, a transmission can interfere only with the transmission of immediate previous, current, and/or immediate next time slot. Hence, there are at most three collision regions given a transmitter as we investigated in section III-B. The three regions for an arbitrary i-th transmitter which has the normalized propagation time distance of α_i are in summary as follows in terms of normalized time distance;

For $R_n(\alpha_i)$ the region for possible collision with the next consecutive time slot:

$$0 \le \tau_n \le \alpha_i - \beta$$

For $R_c(\alpha_i)$ the region for possible collision with the current time slot:

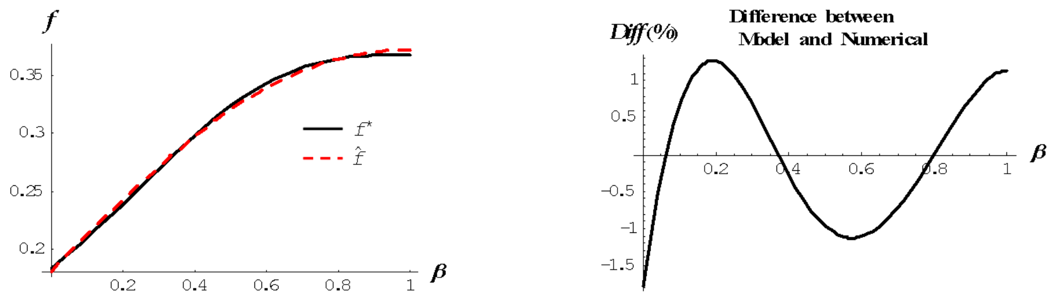
$$0 \le \tau_c \le 1$$

For $R_p(\alpha_i)$ the region for possible collision with the previous consecutive time slot:

$$\alpha_i + \beta \le \tau_p \le 1$$

a	0.2464	p	0.0784	p_1	0.1805
b	−2.9312	q	0.2638	q_1	0.6543
c	−0.9887	r	0.9173	r_1	0.8898

TABLE II
CONSTANTS FOR APPROXIMATION MODELS



(a) (b)

Fig. 4. The maximum number of successful receptions in a time slot: (a) numerically calculated values and its approximation, (b) the difference between numerical values and its approximation

Hence, when β increases, $R_n(\alpha_i)$ decreases monotonically up to $\emptyset$, making the corresponding collision probability monotonically non-increasing; $R_c(\alpha_i)$ stays constant, not changing the probability; and $R_p(\alpha_i)$ decreases monotonically up to $\emptyset$, making the probability monotonically non-increasing. These implies that the probability of no collision for the i-th transmitter $\Pr\{NC|n_i\}$ is monotonically non-decreasing for each i.

Therefore, the expected number of successful receptions in a time slot $f(\beta, \vec{p}) = \sum_i p_i \Pr\{NC|n_i\}$ is monotonically non-decreasing with respect to β.

■

Theorem 2: With the same assumptions of Lemma 1, the maximum expected number of successful packet receptions $f^*(\beta)$ in a time slot over $\vec{p}$ (i.e. $f^*(\beta) = \max_{\vec{p}} f(\beta, \vec{p})$) is monotonically non-decreasing with respect to β. In other words,

$$0 \leq \beta_1 \leq \beta_2 \leq 1 \quad \Rightarrow \quad f^*(\beta_1) \leq f^*(\beta_2)$$

Proof: From the definition of f^* and Lemma 1,

$$f^*(\beta_2) \geq f(\beta_2, \vec{p}) \geq f(\beta_1, \vec{p}), \ \forall \vec{p}$$

Therefore, $f^*(\beta_2)$ is an upper bound of $f(\beta_1, \vec{p})$ for all $\vec{p}$, which implies the following:

$$f^*(\beta_2) \geq \max_{\vec{p}} f(\beta_1, \vec{p}) = f^*(\beta_1)$$

■

Now, we evaluate the maximum numbers of successful packet receptions over the network traffic load λ for 20 values of β starting from 0 to 1 incrementing 0.05 using the numerical method. The black solid line of Figure 4.(a) shows these values connected through interpolation.

As you can see in section III-D, the exact expression of the expected number of successful receptions $f(\beta, \lambda)$ is quite complicated so that it is not quite straightforward to achieve the closed-form expression for its optimization over λ. But, we know from Theorem 2 that the maximized function $f^*(\beta) = \max_\lambda f(\beta, \lambda)$ is monotonically non-decreasing. From this fact and the observation that the log-scale plot of the numerically evaluated $f^*(\beta)$ is approximately of cubic function, we propose the following approximation model for $f^*(\beta)$:

$$\hat{f}(\beta) = e^{a(\beta-1)^2(\beta+b)+c} \tag{19}$$

where a, b, and c are constants and the constraint that $b < -1$ makes sure that the function is monotonically increasing.

The red dash line of Figure 4.(a) is plotted by this approximation with proper constants suggested in Table II acquired through the numerical curve fitting. And Figure 4.(b) shows its accuracy.

B. Maximum Throughput

Now we investigate the maximum throughput S^* over all possible non-negative guard band β and network load λ given the network size τ_m in terms of the maximum propagation delay. Note that it is sufficient to look into only $\beta \in [0,1]$ and $\lambda \in [0,1]$ because $S(\beta,\lambda,t) \leq S(1,1,t)$, $\forall \beta \geq 1$, $\forall \lambda \geq 1$ due to Theorem 4 and Theorem 5.

Theorem 3: Suppose a network of n number of nodes is assumed as that of section III-C with $p = \lambda/n$. Then, the throughput S_n of the PDT-ALOHA protocol with $\lambda \geq 1$ for the network is no higher than when $\lambda = 1$. That is,

$$\lambda \geq 1 \;\Rightarrow\; S_n(\beta,\lambda,t) \leq S_n(\beta,1,t),\; \forall \beta \in [0,1]$$

Proof: The expected number of successful receptions $f_n(\beta,\lambda)$ in a time slot can be expressed as follows using Equation (2), (6), (9), and (10):

$$f_n(\beta,\lambda) = \lambda \int_0^1 2\alpha \Pr\{NC|\alpha\} d\alpha = \lambda \left(1 - \frac{\lambda}{n}\right)^{n-1} \int_0^1 g_n(\beta,\lambda) d\alpha$$

where $g_n(\beta,\lambda)$ is a proper function after extracting $\left(1-\frac{\lambda}{n}\right)^{n-1}$.

Suppose $\lambda \geq 1$. Since $0 < \lambda_1 \leq \lambda_2 < n$ implies $g_n(\beta,\lambda_1) \geq g_n(\beta,\lambda_2)$ for all $\beta \in [0,1]$,

$$\begin{aligned} f_n(\beta,\lambda) &= \lambda \left(1 - \frac{\lambda}{n}\right)^{n-1} \int_0^1 g_n(\beta,\lambda) d\alpha \\ &\leq \lambda \left(1 - \frac{\lambda}{n}\right)^{n-1} \int_0^1 g_n(\beta,1) d\alpha \;\leq\; \left(1 - \frac{1}{n}\right)^{n-1} \int_0^1 g_n(\beta,1) d\alpha \;=\; f_n(\beta,1) \end{aligned}$$

where the last inequality holds since $x(1-x/n)^{n-1} \leq (1-1/n)^{n-1}$ for $\forall x \geq 1$ and $\forall n \geq 2$.

Therefore,

$$S_n(\beta,\lambda,t) = \frac{f_n(\beta,\lambda)}{1+\beta t} \leq \frac{f_n(\beta,1)}{1+\beta t} = S_n(\beta,1,t)$$

■

Theorem 4: Theorem 3 holds for the infinite number of nodes as long as the throughput exists.

Proof: Since $S_n(\beta,\lambda,t) \leq S_n(\beta,1,t)$ for $\forall \lambda \geq 1$ and $\forall n \geq 2$ from Theorem 3,

$$S(\beta,\lambda,t) = \lim_{n\to\infty} S_n(\beta,\lambda,t) \leq \lim_{n\to\infty} S_n(\beta,1,t) = S(\beta,1,t)$$

as long as the limits exist.

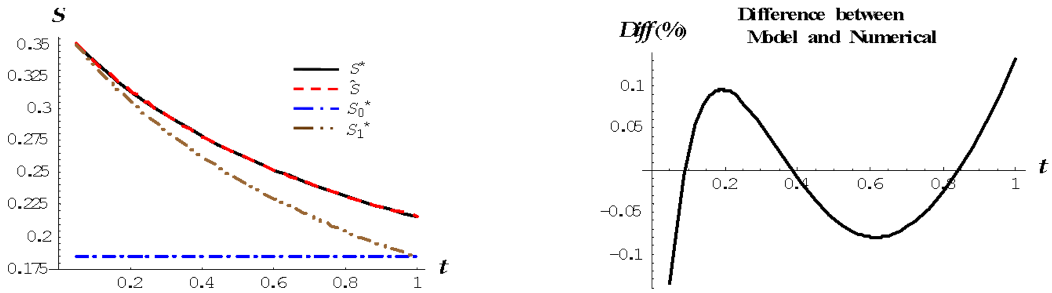
■

Theorem 5: The throughput S with $\beta \geq 1$ of an arbitrary network is no higher than that of $\beta = 1$. That is,

$$\beta \geq 1 \;\Rightarrow\; S(\beta,\vec{p},t) \leq S(1,\vec{p},t)$$

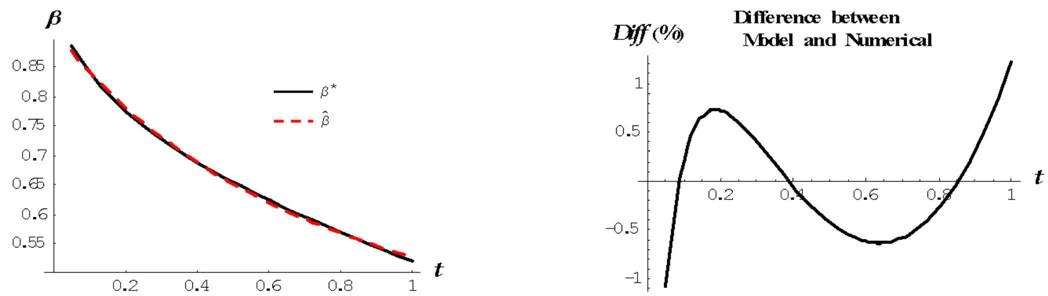
Proof: If $\beta \geq 1$, there is no longer collision of packets between different time slots and β does not have any effect on packets sent in the same time slot. Hence, the expected number of successful packet receptions in a time slot is same for $\beta \geq 1$ as that of $\beta = 1$. However, increasing β makes the size of time slot increases. Therefore, the claim follows.

■



(a) (b)

Fig. 5. The maximum throughput: (a) numerically calculated values and its approximation, (b) the difference between numerical values and its approximation



(a) (b)

Fig. 6. The optimizer β^*: (a) numerically calculated values and its approximation, (b) the difference between numerical values and its approximation

As same as in the previous section, we use the numerical method to model $S^*(t)$. After examining the behavior of S^*, we propose the following simple expression as an approximation of S^*:

$$\hat{S}(t) = p + \frac{q}{t+r} \tag{20}$$

where p, q, and r are constants.

The black solid line of Figure 5.(a) shows the interpolation of 20 data points of the maximum throughput which are found numerically. The red dash line is of our approximation $\hat{S}$ with constants achieved through numerical curve fitting (Table II). And Figure 5.(b) visually shows the accuracy of $\hat{S}$. As you can see, our approximation has reasonably good accuracy.

In order to approximate the optimizer β^*, we propose the following simple expression:

$$\hat{\beta}(t) = p_1 + \frac{q_1}{t+r_1} \tag{21}$$

where p_1, q_1, and r_1 are constants; their proper values are given in Table II through curve fitting.

The black solid line of Figure 6.(a) shows the interpolation of 20 data points of the optimizer which are found numerically. The red dash line is of our approximation $\hat{\beta}$ with constants in Table II. And Figure 5.(b) visually shows the accuracy of $\hat{\beta}$. As you can see, our approximation has reasonably good accuracy.

C. Special Cases

When there is no guard band (i.e $\beta = 0$) or the guard band is full so that there no collision between packets from different time slots (i.e. $\beta = 1$) we have a closed-form expression for throughput which is simple enough to analyze analytically the maximum throughput. When $\beta = 0$ it is easy to see from Equation (6), (14), and (15) that the throughput is as follows:

$$S_0(\lambda) \doteq S(\beta = 0, \lambda, t) = \lambda e^{-2\lambda} \tag{22}$$

There is no guard band in this case which makes the maximum propagation time irrelevant to the throughput, which Equation (22) confirms. Note that the throughput in this case become the throughput of the classical unslotted ALOHA protocol [13], [15] as pointed by [17], [16].

The maximum throughput can be obtained simply using the derivative since S_0 is convex. The maximum is achieved at $\lambda = 0.5$ as follows:

$$S_0^* = e^{-1}/2 \tag{23}$$

When $\beta = 1$ the throughput is as follows from Equation (6), (14), and (16):

$$S_1(\lambda, t) \doteq S(\beta = 1, \lambda, t) = \frac{\lambda e^{-\lambda}}{1+t} \tag{24}$$

Because S_1 is convex regarding λ at any $t \in (0, 1]$, we can obtain its maximum given t using the partial derivative as follows:

$$S_1^*(t) = \frac{e^{-1}}{1+t} \tag{25}$$

where the maximizer is $\lambda = 1$.

Figure 5.(a) shows S_0^* and S_1^* as well; it can be seen that both of them are suboptimal although S_1^* approaches the maximum as t goes to 0. When $t = 1$, the maximum propagation delay is equal to the transmission time of a packet; the throughput becomes same whether PDT-ALOHA has the full guard band or no guard band at all.

V. Acknowledgement

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VI. Conclusion

We analyze mathematically the performance of the PDT-ALOHA protocol in this work. We investigate different metrics of performances – expected number of successful packet receptions in a time slot, throughput, maximum throughput. We obtain exact expressions for the number of receptions and throughput in terms of well-known functions. Although the exact expressions are quite complicated, it is fairly fast to numerically calculate them with given parameters. Further, we obtain very simple expressions for the maximum throughput which are shown to be very good approximations.

We also prove a number of interesting and useful properties concerning the performance of the PDT-ALOHA protocol. We prove that the expected number of successful packet receptions is independent of the propagation speed; that its maximum is non-decreasing as the size of guard band increases; and derive a bound on the network load that offers the maximum throughput.

In the future, we would like to extend these results for 3-dimensional networks because nodes are more likely to be deployed in the 3-dimensional area under the water unlike on the ground.

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APPENDIX

A. Proof of Uniform Convergence

Before proving the theorem, we first state the following facts and prove the subsequent lemmas.

Fact A.1: Suppose K is compact, and

(a) $\{f_n\}$ is a sequence of continuous functions on K,

(b) $\{f_n\}$ converges pointwise to a continous function f on K,

(c) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K, n = 1, 2, 3, \ldots$.

Then, $f_n \to f$ uniformly on K.

Reference: Theorem 7.13 in page 150 of [14].

Fact A.2: If $\{f_n\}$ and $\{g_n\}$ converges uniformly on a set E and they are sequences of bounded functions, then $\{f_n g_n\}$ converges uniformly on E.

Reference: Page 165 of [14].

Lemma A.2: Suppose

$$f_n(x) = \left(1 - \frac{x}{n}\right)^{n-1}$$
$$f(x) = e^{-x}$$

Then the sequence of functions $\{f_n\}, n = 2, 3, \ldots$, converges uniformly on $x \in [0, 1] \subset \mathcal{R}$ to f.

Proof: Let $X = [0, 1] \subset \mathcal{R}$. From Fact A.1, what we need to show are (i) $f_n(x)$ is continuous on X for all n, (ii) $\{f_n\}$ converges pointwise to a continuous function f on X, (iii) $f_n(x) \geq f_{n+1}(x)$ for all $x \in X, n = 2, 3, 4, \ldots$.

It is easy to see that $f_n(x)$ and $f(x)$ are continuous on X for all n and that $\lim_{n\to\infty} f_n(x) = f(x)$.

Suppose $n \in \{r : r \geq 2, r \in \mathcal{R}\}$. Then,

$$\tilde{f}_n(x) = \frac{\partial f_n(x)}{\partial n} = \left(1 - \frac{x}{n}\right)^{n-1} \left\{ \frac{(n-1)x}{n(n-x)} + \ln\left(1 - \frac{x}{n}\right) \right\}$$

Let

$$g_n(x) = \frac{(n-1)x}{n(n-x)} + \ln\left(1 - \frac{x}{n}\right)$$

Then, $g_n'(x) = \frac{x-1}{(n-x)^2}$. Hence, $g_n(x)$ monotonically decreasing on $[0, 1]$ for all n, which implies that, with the fact that $g_n(0) = 0$, $g_n(x) \leq 0$ for all $x \in [0, 1]$ and all n. Hence, $\tilde{f}_n(x) \leq 0$ for all $x \in X$ and all n implying that

$f_n(x)$ is monotonically non-increasing as n increases for all $x \in X$. Therefore, it follows that $f_n(x) \geq f_{n+1}(x)$ for all $x \in X$ and all $n = 2, 3, 4, \ldots$.

■

Theorem A.6: The integrand of Equation (6) converges uniformly on $[0, 1]$ if $p = \frac{\lambda}{n}$ where $0 \leq \lambda \leq 1$.

Proof: From Fact A.2, it is sufficient to show that each of terms $\alpha \Pr\{NC|\alpha\}$ with Equations (11) and (12) converges uniformly on its domain of α and it is a sequence of bounded functions on $A = [0, 1]$.

First, let us show that the term $(1 - \frac{\lambda}{n} + \frac{\lambda}{n}(\alpha + \beta)^2)^{n-1}$, denoted by $h_n(\alpha)$, converges uniformly on $\alpha \in [0, \beta]$, which is from the case where $0 \leq \alpha \leq \beta \leq 0.5$. Because the range $\mathfrak{R}(\lambda(1 - (\alpha + \beta)^2))$ is a compact set in A and $h_n(\alpha)$ can be rewritten as

$$h_n(\alpha) = \left(1 - \frac{\lambda(1 - (\alpha + \beta)^2)}{n}\right)^{n-1}$$

its uniform convergence follows from Lemma A.2. And $|h_n(\alpha)| \leq 1$ for all α and all n.

In the similar way, it can be proven that other terms satisfy the conditions without difficulty. Therefore, the claim follows.

■