TDMA scheduling feasibility of the Receiver Capacity Model

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1 Introduction

The receiver capacity model uses local constraints to define the achievable rate region of a given routing topology. In this report we show that for all rate vectors, that satisfy the receiver capacity constraints with the receiver capacity set to $\frac{1}{3}$, there exists a feasible TDMA schedule that can satisfy these rate vectors.

2 The Receiver Capacity Model

In this section we present an example to illustrate the receiver capacity model over a collection tree.

Figure 1 shows a 6 node topology. The solid lines indicate a parent child relationship in the tree. The dashed line represent noise links. For each source, any rate consumed by the source on the link with its parent will result in consumption of an equal rate on the noise links. Thus, when node 2 sends its data to node 1, node 2 not only consumes capacity at node 1 but also at node 3, since the same flow exists over link $2 \rightarrow 1$ and noise link $2 \rightarrow 3$.

The radios are assumed to be half duplex. The half duplex nature of the radio forces flows to be received at a particular rate in a particular slot and then forwarded at the same rate in the next available slot. This results in flows originating from a child consuming twice the allocated rate at the parent.

The receiver capacity constraints on the rates at node 3 will be as follows:

$$r_2^{noise} + r_3^{noise} + r_6^{src} \leq B_3 \tag{1}$$

where B_3 is the receiver capacity of node 3 and r_6^{src} is the source rate of node 6. r_2^{noise} and r_3^{noise} are the output rates at node 2 and node 3 respectively and are given by:

$$r_2^{noise} = r_2^{src} + r_4^{src} + r_5^{src}$$

and

$$r_3^{noise} = r_3^{src} + r_6^{src}$$

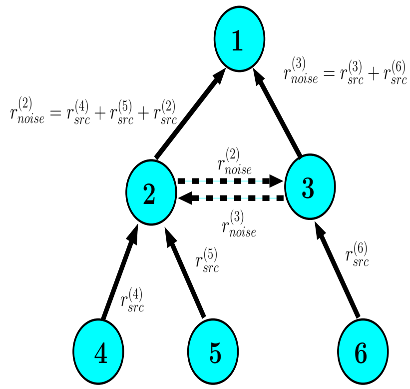

Figure 1: A 6 node topology: An illustrative example of the receiver capacity model

The half duplex assumption for the radios forces the term r_6^{src} to appear twice in equation 1. Once independently to account for the consumption of bandwidth during reception at node 3 and once as part of the term r_3^{noise} to account for the forwarding of the flow originating at node 6.

In general the receiver capacity constraint at a node i can be given as follows:

$$\sum_{j \in C^{(i)}} r_j^{src} + \sum_{j \in N_i} \sum_{k \in C_j} r_k^{src} + \sum_{j \in N_i} r_j^{src} \leq B_i \tag{2}$$

Where N_i is the set of all neighbors of i. The half duplex assumption implies that $i \in N_i$. C_i is the set of all nodes j that have i in its path to the sink. r_j^{src} represents the rate at which data generated at node j is being transmitted.

Although the above example was specific to a tree, the receiver capacity model is applicable to a directed graph as well.

The constraints presented above were flow constraints. The flow constraints of the receiver capacity model can be written purely as link constraints as follows: For node 1 the constraint will be:

$$f_{21} + f_{31} \leq B_1$$

The constraint at node 2 will be:

$$f_{21} + f_{42} + f_{52} + f_{31} \leq B_2$$

Where f_{ij} is the link rate on link ij, with i as the transmitter and j as the receiver. In order to match the flow constraints of the receiver capacity model we require to at least these extra constraints, in addition to the link constraints presented above:

$$f_{42} \leq f_{21}$$
$$f_{52} \leq f_{21}$$
$$f_{63} \leq f_{31}$$

Due to these extra constraints, the flow constraints will always be more constrained than the link constraints in a receiver capacity model(at least over a collection tree). Thus, if we are able to develop a sufficiency constraint for the link rates over a receiver capacity model, for a collection tree the sufficiency condition will hold for the flow rates as well.

3 Preliminaries

In this section we present a quantitative description of a general communication graph and various terms pertaining to this graph that will be used in our proof for the sufficiency condition of the link rates for the receiver capacity model.

We represent the nodes in the network, and possible communication, with a directed labeled graph $G = (V, L, f, w)$ where V represents the set of nodes in the network and L the set of edges (links) in the network. Each link $l \in L$ is associated with two labels f and w. The label $f(l)$ represents a demand rate at which data needs to be transmitted over the link l. The label $w(l)$ represents the number of slots that are required in order to transmit the data over the link at a rate $f(l)$.

Over the graph G only a subset of the link $l \in L$ are active. This subset is determined by the transmitter receiver pairs that want to exchange data. If l is part of the set of links that are active then $f(l) \geq 0$, else $f(l) = 0$. It is assumed that $f(l), \forall\, l \in L$, are rational. The value of $w(l)$, associated with each link $l \in L$ is determined as follows:

$$w(l) = \frac{f(l)}{\tau}$$

Where τ is some rational number. We can find a $w(l), \forall\, l \in L$, since $f(l)$ is assumed to be rational. The value τ can be thought of as the slot interval. Thus, we have a total of $\frac{1}{\tau}$ slots per second to schedule, and a link requiring a rate of $f(l)$ will require $w(l) = \frac{f(l)}{\tau}$ slots per second to achieve this rate. There can be multiple such τ values that will give an integral value for all links $l \in L$, we choose the largest of these τ that makes $w(l)$ a positive integer $\forall\, l \in L$. Thus, every link in the graph G will have two labels associated with it, a rate label $f(l)$ and a slot label $w(l)$.

Our objective is to determine a sufficiency condition such that, given a demand vector f that satisfies the sufficiency condition, it is possible to design a TDMA schedule that can satisfy the demand f. Some other variables that are defined for the graph G are as follows:

- $r(l)$ is defined as the receiver of a link $l \in L$.
- $t(l)$ is defined as the transmitter of a link $l \in L$.
- N_i, the set of neighbors of node i, such that for any $j \in N_i$ there exists a link $l \in L$ such that $r(l) = i$, $t(l) = j$ and $f(l) = 0$. *NOTE: Under the receiver capacity model such a link will be a noise link.*

NOTE: We assume that each receiver i in the graph G has a receiver capacity of 1. We have set the receiver capacity to 1 to keep the analysis tractable. We believe the results presented here can be extended to cases where receiver capacities are greater then 1 by changing the calculation of the number of slots required to schedule a given demand vector.

We now define a few terms pertaining to the graph G.

Definition 1. *An interference set $I(i)$ for a node is defined as $I(i) = \{l : r(l) = i, \text{ or } t(l) = i, \text{ or } t(l) = j, \ j \in N_i\}$. The set $I(i)$ thus contains the set of links that have i as a receiver, the set of links that have i as a transmitter, and the set of links that have a transmitter who is a neighbor of i.*

NOTE: *The cardinality of the interference set $I(i)$ at a node i represents the total number of terms in the receiver capacity constraint at node i, where $\alpha \leq 1$.*

Definition 2. *The graph G α-satisfies the receiver capacity model (RCM) if :*

$$\sum_{l \in I(i)} f(l) \leq \alpha, \forall \ i \tag{3}$$

NOTE: *Equation 3 is the receiver capacity constraint at node i.*

Definition 3. *We define the contention graph G_c for the graph G, by replacing every edge $l \in L$ with $w(l)$ vertices in G_c. The $w(l)$ vertices in G_c form a clique. Beyond this, we add an edge between two vertices in G_c if the edges of the corresponding vertices in G interfere with each others transmissions. The vertices in the graph G_c form the set V_c, and the edges in the graph G_c form the set E_c. The maximum clique size in G_c is denoted by Δ_c.*

Definition 4. *We define the contention graph G_{dc} as a directed version of the graph G_c. The set of vertices of the graph G_{dc}, $V_{dc} = V_c$. The set of edges E_{dc} is a directed version of the edges in E_c. A directed edge $e_{dc} \in E_{dc}$, from $v_i \in V_{dc}$ to $v_j \in V_{dc}$ exists if the transmitter of the corresponding link $l_i \in L$ interferes with the reception of $l_j \in L$. A reverse directed edge will exist if the transmitter of $l_j \in L$ interferes with the reception of l_i.*

Definition 5. *We define the maximum interference set, Δ_I for the graph G as follows:*

$$\Delta_I = \max_{\forall \ i \ \in V} \left(\sum_{l \in I(i)} w(l) \right)$$

4 The sufficiency condition

The answer to the question of the sufficiency condition which guarantees the schedulability of the demand vector f is given by the following theorem:

Theorem 1. *A graph G that is 1/3-satisfying RCM can be feasibly TDMA scheduled.*

The proof of this theorem follows by combining the following theorems 2 and 3. We first present and prove them before proving this result.

Theorem 2. *A graph G, that is α-satisfying cannot be scheduled in less than $\frac{\Delta_I}{\alpha}$ slots.*

Proof. A graph G that is α-satisfying is given by :

$$\sum_{l \in I(i)} f(l) \leq \alpha, \forall\ i$$

Multiplying both sides by $\frac{1}{\tau}$ we have:

$$\sum_{l \in I(i)} \frac{f(l)}{\tau} \leq \frac{\alpha}{\tau}, \forall\ i$$

The L.H.S can be rewritten in terms of $w(l)$ as:

$$\sum_{l \in I(i)} w(l) \leq \frac{\alpha}{\tau}, \forall\ i$$

By definition 2 we have:

$$\Delta_I \leq \frac{\alpha}{\tau}$$

This implies that

$$\frac{\Delta_I}{\alpha} \leq \frac{1}{\tau}$$

$\frac{1}{\tau}$ represents the total number of slots available per second. Thus, a graph G, that is α-satisfying cannot be scheduled in less than $\frac{\Delta_I}{\alpha}$ slots. $\square$

Theorem 3. *A graph G can be scheduled in at most $3\Delta_I$ slots.*

We will require lemma 1 and 2, presented below, to prove this theorem. We therefore present these lemma's before we prove this theorem.

Lemma 1. *The maximum clique size of the contention graph G_c is no more than twice the maximum interference set Δ_I for the graph G. i.e.*

$$\Delta_I \geq \frac{\Delta_c}{2}$$

Proof. The interference set $I(v_i)$, of a vertex $v_i \in V$ in the graph G and the in-degree $D_{v_j}^{in}$ for the vertex $v_j \in V_{dc}$ of the directed contention graph G_{dc} such that $r(e_j) = v_i, e_j \in V$, are related as follows:

$$|I(v_i)| = D_{v_j}^{in} + 1$$

Thus, $\Delta_I = \max_{v\ \in\ V_{dc}} (D_v^{in}) + 1$.

We claim that $\max_{v \,\in\, V_{dc}} (D_v^{in}) + 1 \geq \frac{\Delta_c}{2}$.

To prove the above claim assuming $\max_{v \,\in\, V_{dc}} (D_v^{in}) + 1 < \frac{\Delta_c}{2}$.

Consider the set of vertices forming the maximum clique in G_c and call this set of vertices the set C. Since $V_{dc} = V_c$, the set C exists in G_{dc} as well. From our assumption, for the graph G_{dc}, the in-degree of each vertex $v \in C$ is less than $\frac{1}{2}n - 1$, where $\Delta_c = n$. Thus for these set of vertices the total number of incoming edges is less than $n(\frac{1}{2}n - 1)$ which is equal to $\frac{1}{2}n^2 - n$.

However the total number of incoming edges, which is equal to the total number of edges for this set of vertices belonging to C is $\frac{n^2}{2} - \frac{n}{2}$. Since $\frac{1}{2}n^2 - n < \frac{n^2}{2} - \frac{n}{2}$, it leads to contradiction.

Hence $\max_{v \in V_{dc}}(D_v^{in}) + 1 \geq \frac{\Delta_c}{2}$ and hence $\Delta_I \geq \frac{\Delta_c}{2}$. □

Lemma 2. *The number of slots required to achieve a feasible schedule in the graph G is no greater than $\frac{3}{2}\Delta_c$.*

Proof. We first define a graph H, such that edges on this graph are vertices on G_c and nodes in this graph are edges on G_c. Since G_c is the line graph of H, edge coloring of H is vertex coloring of G_c. If Δ_H is the maximum node degree in H, then as shown by Shannon in [1] the maximum number of colors that will be required to edge color the graph H will be $\frac{3}{2}\Delta_H$. However since G_c is the line graph of H by construction $\Delta_c \geq \Delta_H$ and hence H can be edge colored in $\frac{3}{2}\Delta_c$ colors.

The bound on the number of colors required for vertex coloring of the graph G_c, gives us a bound on the number of slots required for a feasible schedule in G. This statement is true since no two vertices in G_c, which share an edge can have the same color in a feasible vertex coloring. Also, if two vertices in G_c share an edge, the corresponding links in G cannot be scheduled simultaneously. Therefore, interchanging a color for a slot, a feasible vertex coloring on G_c also represents a feasible TDMA schedule for G satisfying a demand vector f. Thus, the number of slots required to achieve a feasible schedule in the graph G is no greater than $\frac{3}{2}\Delta_c$. □

We can now prove theorem 3.

Proof. **Theorem 3**
By lemma 2, since the graph G can be scheduled in $\frac{3}{2}\Delta_c$ slots this implies that the graph G can be scheduled in $3\Delta_I$ slots as well. This is because by lemma 1 $\Delta_I \geq \frac{\Delta_c}{2}$.

□

Given the proofs of theorems 2 and 3, we can now prove theorem 1.

Proof. **Theorem 1**
Theorem 3 states that if we have $3\Delta_I$ slots, we can guarantee a schedule for the scenario represented by graph G. Theorem 2 states that the minimum

number of slots required by an α-satisfying demand vector f is $\frac{\Delta_f}{\alpha}$. It can be easily seen that by setting $\alpha = \frac{1}{3}$ the minimum number of slots required to TDMA schedule a demand vector f matches the number of slots that guarantee a feasible schedule for the graph G. This implies that for $\alpha \leq \frac{1}{3}$ all demand rate vectors f, that are α-satisfying , can be guaranteed a feasible TDMA schedule. This proves our main theorem. $\square$

5 Related Work

The proof for the sufficiency condition guaranteeing schedulability of a demand vector has been built upon ideas from two works. The work by Fang and Bensaou [2] states without proof that it can be shown for the clique capacity model that a bandwidth constraint of $\frac{2}{3}$ yields a feasible TDMA schedule. We use this result, essentially proving it in lemma 2. The techniques of using bounds on edge coloring to prove bounds on the number of time slots is similar to the work by Kodialam and Nandagopal [3]. Kodialam and Nandagopal [3] show the sufficiency condition for half-duplex systems with linear edge-rate constraints where interference between non adjacent transmission links is eliminated through the use of orthogonal channels.

References

[1] Shannon, C.E., "A Theorem on Coloring the Lines of a Network", J. of Math. Physics, 28, pp. 148-151, 1949.

[2] Zuyuan Fang and Brahim Bensaou, "Fair Bandwidth Sharing Algorithms based on Game Theory Frameworks for Wireless Ad-hoc Networks," In. Proc. of INFOCOM 2004.

[3] Murali Kodialam and Thyaga Nandagopal, "Characterizing Achievable Rates in Multi-hop Wireless Mesh Networks with Orthogonal Channels," IEEE/ACM Transactions of Networking, Aug 2005