Sub-Carrier Allocation in OFDM Systems: Complexity, Approximability and Algorithms

Pai-Han Huang, Yi Gai, Bhaskar Krishnamachari
EE-System
University of Southern California
{paihanhu,ygai,bkrishna}@usc.edu

Ashwin Sridharan
Sprint Lab
Ashwin.Sridharan@sprint.com

Abstract

Orthogonal Frequency Division Multiplexing (OFDM) has become the de facto standard for fourth generation wireless networks[5]. In such a network, the frequency band is divided into numerous orthogonal sub-carriers. In each time-slot, disjoint sets of sub-carriers can be assigned to users based on some target objective. The users in turn transmit data by spreading the information across the assigned sub-carriers. The main contribution of this work is a formal analysis of the complexity of this class of resource allocation problems for various objectives and scenarios. Specifically we formally prove that the sub-carrier resource allocation problem is NP-hard for both power minimization as well as rate maximization in both uplink and downlink scenarios. More importantly, we also provide in-approximability results for these scenarios, outlining a hard limit on what deterministic allocation algorithms can achieve for this class of problems. While these results are mostly of a negative nature, we also propose a class of heuristics, called k-interchange, which are shown to yield close to optimal performance via simulations.

1. Introduction

Orthogonal Frequency Division Multiple Access (OFDMA) is a time-frequency hybrid system wherein the frequency band is divided into a large number of small bands called *sub-carriers* that use specific frequencies so as to be completely orthogonal to each other. In every time-slot, each user is assigned a disjoint set of sub-carriers across which the user may spread information for transmission purposes. Because of its capability of exploiting multi-path fading and spatial/temporal diversity to improve performance, it becomes physical layer transmission scheme of choice adopted by fourth generation wireless networks, e.g. cellular networks [5], broadband LANs[6], in order to provide high-speed mobile wireless data services.

Although the OFDMA framework provides a mechanism for a user to spread information across the set of assigned sub-carriers, it still leaves the question of how to assign sub-carriers to users. This immediately presents itself as a resource allocation problem in combinatorics: in each time-slot, give m users and n sub-carriers, how to assign disjoint sets of sub-carriers to each user so as to optimize some system metric. The allocation problem has received active interest in the research community and has been studied from basically two perspectives (based on the objective function): schemes that seek to minimize the amount of transmit power [8][17] or those that seek to achieve maximum throughput [10][14]. However, a common theme (elaborated in Section 2) across these works is that they all propose heuristics that are evaluated only through simulations. This leaves open the question of how well these algorithms perform in general, and at a more fundamental level, how well can *any* deterministic algorithm perform for this class of problems.

We address this gap in existing literature by formally addressing the complexity of the sub-carrier allocation problem under several scenarios, namely uplink and downlink OFDMA, for both power minimization and rate maximization. The problem is shown to be NP-hard in all versions. While this is of academic interest, more importantly we also present approximability bounds for these problems, that is, we bound the best performance achievable by any algorithm. In addition, we propose a *k-interchange* heuristic which allows a natural trade-off between time complexity and performance and utilize it to demonstrate both: its near-optimal performance in simulations as well as worst case scenarios.

Our contributions can be summarized as follows :

1. We formally show that the general problem of sub-carrier allocation with the objective of power minimization is NP-hard and cannot be approximated

within a fixed factor by any deterministic algorithm.

2. We also show that the rate-maximization version is NP-hard and cannot be approximated to a factor better than $\frac{m}{m+1}$.

3. We propose a local search based, polynomial time algorithm, $k-$interchange. In most scenarios, $k-$interchange demonstrates close to optimal performance. On the other hand, we also use the k-interchange heuristic to demonstrate scenarios corresponding to when worst case performance can occur.

This paper is organized as follows: In Section 2 we outline past related work and differentiate our contribution. Section 3 presents the system model and formal problem formulation. In Section 4, we prove the hardness and inapproximability of our problem. Section 5, presents numerical comparison of our proposed algorithms and the optimal solutions and the worst case scenario. Future directions are addressed in Section 6.

2. Related Work

Although the continuous case of single-user/multi-subcarrier has been optimally solved [4][2], and the discrete version[7] of single-user/multi-subcarrier has also been solved near-optimally, the case of multi-user/multi-subcarrier power allocation is still an area of active investigation. In addition, it has been shown that adaptive resource allocation can significantly increase the capacity of OFDM systems[16], comparing with fixed resource allocation strategy [13][12][3]. Thus, how to allocate power or data rate on each sub-carrier to satisfy all requests, while optimizing either power or throughput related objectives becomes a popular research issue.

Wong *et al.*[17] propose a multi-user, multi sub-carrier, bit and power allocation algorithm, which aims to minimize the overall power consumption. Their iterative search algorithm exploits Lagrangian Relaxation technique. However, this algorithm does not converge rapidly in general. Rhee *et al.*[11] try to maximize the minimal user throughput, such that a fixed total power budget is given. One of their assumption: every sub-carrier is allocated with equal power, limits the applicability of their solution. Kivanc *et al.*[8] adopt a similar formulation as [17]. The authors propose a two-step solution: first determine the number of sub-carriers allocated to each user, and then allocate sub-carriers to users in a greedy manner. Thereafter, they refine their solution quality by using local search technique. Alen *et al.*[1] devise a distributed algorithm aims to maximize system capacity. They first divide available sub-carriers into a set of partitions, and let users contend these partitions themselves. To solve the issue of partition sharing, i.e. multiple users access the same partition, their solution includes a conflict resolving mechanism.

In sum, none of existing papers we are aware of has a proof of hardness or performance bound about the sub-carrier allocation problem.

3. System Model and Problem Formulation

We assume complete knowledge of channel states at both transmitter and receiver ends, and they do not change during the scheduling. A subcarrier can be used by at most 1 user at any instance, and every user's rate request must be satisfied. We refer these two restrictions to "feasibility constraints".

We assume that the system has m sub-carriers and a static population of n users. The focus of this work is allocation of sub-carriers in a single time-slot (a granularity readily available in current 4G networks). Each user i requests a rate r_i in the time-slot. Sub-carrier allocation is determined at the base-station as a function of the perceived signal strength of each user on each sub-carrier as well as the user requested rate. In the uplink environment, after the allocation is conveyed to the user (we assume the implicit presence of such a mechanism. For example, in WiMax, this information is conveyed using a UL-MAP message), each user performs rate loading across the assigned sub-carriers to transmit information. On the other hand, the rate loading becomes a task of the base-station in the down-link environment. Rate loading is typically governed by the available transmission rates (i.e., modulation schemes).

For convenience, we summarize the symbols used in this paper as follows.

- S: The set of all sub-carriers. $S = \{s_i | i = 1, \ldots, m\}$.
- U: The set of all users. $U = \{u_i | i = 1, \ldots, n\}$
- r_i: The constant data rate that the user i requests.
- r_{ij}: The data rate user i loads on subcarrier j.
- p_i: The power budget of user i.
- S^i: This represents the set of sub-carriers which are allocated to user i.
- $f_{ij}(r_{ij})$: The power required to transmit at rate r_{ij} by user i on sub-carrier j is given by $f_{ij}(r_{ij})$.
- $d_{i,j}$: $d_{i,j} = 1$ if sub-carrier j is allocated to user i. Otherwise, it equals 0.
- T_i: In practice, a user is allowed to transmit only at certain discrete rates. We assume that the set of feasible rates allowed on a sub-carrier i is given by T_i.

3.1 Uplink Sub-carrier Allocation

We assume the functions $\{f_{ij}(\cdot)\}$ are convex, increasing, continuous, and $f_{ij}(0) = 0, \forall u_i \in U, s_j \in S$. In addition, every user i has a power limit p_i that needs to be respected. The objective of our interest is to minimize the maximal individual user's power consumption, and the problem is referred to *Continuous Uplink Sub-carrier Allocation*. Mathematically, we can formulate it as Equation 1:

$$\begin{aligned}
\text{minimize} \quad & max_{u_i \in U}\{\sum_{s_j \in S} f_{ij}(r_{ij}d_{i,j})\} \\
\text{subject to:} \quad & \sum_{s_j \in S} r_{ij}d_{i,j} = r_i, \forall u_i \in U \\
& d_{i,j} \in \{0,1\} \forall u_i \in U, s_j \in S \\
& \sum_{u_i \in U} d_{i,j} \leq 1, \forall s_j \in S \\
& r_{ij} \geq 0, \forall u_i \in U, s_j \in S \\
& \sum_{s_j \in S} d_{i,j} \geq 1, \forall u_i \in U \\
& \sum_{s_j \in S} f_{ij}(r_{ij}d_{i,j}) \leq p_i, \forall u_i \in U \qquad (1)
\end{aligned}$$

If $\{f_{ij}(\cdot)\}$ are convex, increasing, $f_{ij}(0) = 0, \forall u_i \in U, s_j \in S$, but discrete, i.e. only certain rate are allowed on a sub-carrier, then this problem is referred to *Discrete Uplink Sub-carrier Allocation*, and can be mathematically stated by replacing $r_{ij} \geq 0$ with $r_{ij} \in T_j$ in Equation 1.

Another interesting uplink problem is trying to maximize total transmission rate, subject to a set of fixed power budget, and can be mathematically described as Equation 2:

$$\begin{aligned}
\text{maximize} \quad & \sum_{u_i \in U}\sum_{s_j \in S} r_{ij}d_{i,j} \\
\text{subject to:} \quad & \sum_{s_j \in S} f_{ij}(r_{ij}d_{i,j}) \leq p_i, \forall u_i \in U \\
& d_{i,j} \in \{0,1\}, \forall u_i \in U, s_j \in S \\
& \sum_{u_i \in U} d_{i,j} \leq 1, \forall s_j \in S \\
& \sum_{s_j \in S} d_{i,j} \geq 1, \forall u_i \in U \qquad (2)
\end{aligned}$$

Similarly, if the given rate-power equations are discrete, then we add $r_{ij} \in T_j, \forall u_i \in U, s_j \in S$ into Equation 2.

We refer the above two versions of rate maximizing problems as *Continuous Rate Maximizing Sub-carrier Allocation* and *Discrete Rate Maximizing Sub-carrier Allocation*, respectively.

3.2 Downlink Sub-carrier Allocation

When OFDM downlink sub-carrier allocation is considered, a key difference is that individual user power constraints are replaced with the power constraint for total sum of transmit powers, i.e the entire base-station's transmit power (which is critical in practice to reduce inter-cell interference). This problem is referred to *Continuous Downlink Sub-carrier Allocation*, and this optimization problem can be described by removing the last constraint, the power budget constraint, in Equation 1, and replacing the objective with minimize $\sum_{u_i \in U}\sum_{s_j \in S} f_{ij}(r_{ij}d_{ij})$. In addition, if $\{f_{ij}(\cdot)\}$ are discrete, then this problem is referred to *Discrete Downlink Sub-carrier Allocation*, and we have to replace one more constraint $r_{ij} \geq 0$ with $r_{ij} \in T_j$.

4. NP-hardness and In-approximability

4.1 NP-hardness Proof

The decision version of Continuous Downlink Sub-carrier Allocation can be stated as:

Problem 1 Given a set of n user rate requests, convex continuous rate-power equations for n users and m sub-carriers combination, and $m > n$, does there exist a set of sub-carrier and user assignment with rate allocation on individual sub-carrier, such that the sum of all user's power consumption is at most P, every user's request is satisfied, and no sub-carrier is allocated to more than 1 user?

The NP-hardness of Continuous Downlink Sub-carrier Allocation is shown below.

Theorem 1 *Continuous Downlink Sub-carrier Allocation is NP-hard.*

proof: We first show that Subset Sum[9] $\leq_P$ Continuous Downlink Sub-carrier Allocation. Consider an arbitrary instance of Subset Sum, with a set of natural numbers $W = \{w_i\}$, and a target V. We construct a corresponding two user Continuous Uplink Sub-carrier Allocation instance as follows. For every w_i, we construct a sub-carrier i with a rate-power equation, which is the same for both users, i.e. $f_{1i} = f_{2i} = f_i$. Specifically, f_i has the following properties: the power for rate 0 is 0, the power of rate w_i is no more than $\frac{P}{|W|}$, the power of rate $w_i + \frac{1}{|W|}$ is no less than P, and the rate-power relation between rate 0 and infinity is continuous and convex. This constructed rate-power equation can be plotted as Figure 1. To satisfy the above constraints for a constructed rate-power equation, we can use a function like $f_i(r) = \alpha(2^{\beta r} - 1)$ where α and β are unknown, or a piece-wise linear function. In addition, finding feasible α and β, or a piece wise linear function can be done in polynomial time.

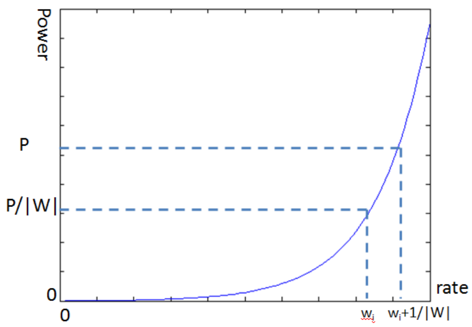

Figure 1. An example rate-power curve of a constructed sub-carrier in Continuous Downlink Sub-carrier Allocation. (Note: This figure is only for illustration purpose.)

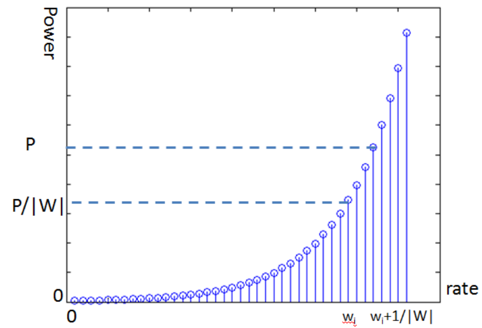

Figure 2. An example rate-power curve of a constructed sub-carrier in Discrete Uplink Sub-carrier Allocation. (Note: This figure is only for illustration purpose.)

We claim that Subset Sum has a satisfying solution if and only if Continuous Downlink Sub-carrier Allocation has an assignment, which can satisfy two users with rate requests V and $(\sum_{i=1}^{m} w_i - V)$ and the total power consumption is no more than P. Suppose Subset Sum has a solution such that the sum of the subset S is exactly V. If we allocate every corresponding sub-carrier in S to one user, and all the remaining sub-carriers to the other user, and load every sub-carrier with rate w_i, then this assignment would be a satisfying solution to Continuous Downlink Sub-carrier Allocation. On the other hand, if Subset Sum is a NO instance, then no subset can give us a sum exactly V. Because every w_i is a natural number, the difference of sum between any subset and V must be no less than 1. In addition, the largest number of sub-carrier a user can be allocated is $|W| - 1$. Consequently, one of the two users has to load at least one of the sub-carrier assigned to him/her with rate higher than $w_i + \frac{1}{|W|}$, which implies the maximal individual power consumption higher than P, thus making the total power consumption also higher than P.

In the proof of Theorem 1, if we replace the rate-power equations with discrete ones, which can be obtained by discretizing the continuous counterpart with polynomially many pieces as plotted in Figure 2, then a line by line similar proof can give us the following conclusion.

Theorem 2 *Discrete Downlink Sub-carrier Allocation is NP-hard.*

Next, we prove the NP-hardness of Continuous Uplink Sub-carrier Allocation. Note that, due to the space constraint and the similarity between Problem 1, the decision problem statement is omitted.

Theorem 3 *Continuous Uplink Sub-carrier Allocation is NP-hard.*

proof: The proof of Subset Sum $\leq_P$ Continuous Uplink Sub-carrier Allocation is line by line similar to Theorem 1 with two additional conditions: the power budget $p_1 = \infty$ and $p_2 = \infty$.

We claim that Subset Sum has a satisfying solution if and only if Continuous Uplink Sub-carrier Allocation has an assignment, which can satisfy two users with rate requests V and $(\sum_{i=1}^{m} w_i - V)$ and the maximal individual power consumption is no more than P. If Subset Sum has a satisfying solution, then the optimal solution for Continuous Uplink Sub-carrier Allocation has value no more than P, thus making it a YES instance. On the other hand, if Subset Sum is a NO instance, then, for the same reasons as in the proof of Theorem 1, we know the maximal individual power consumption is higher than P, which implies a NO instance of Continuous Uplink Sub-carrier Allocation.

Also, by using sub-carriers with rate-power relations as plotted in Figure 2, a line by line similar proof as in Theorem 3 can give us the following conclusion.

Theorem 4 *Discrete Uplink Sub-carrier Allocation is NP-hard.*

The decision version of Continuous Rate Maximizing Sub-carrier Allocation is stated as follows.

Problem 2 Given a set of n user power budgets, convex continuous rate-power equations for n users and m sub-carriers combination, and $m > n$, does there exist a set of sub-carrier and user assignment with power allocation on individual sub-carrier, such that the sum of all user's rate is at least R, every user's power budget is honored, and no sub-carrier is allocated to more than 1 user?

Theorem 5 *Continuous Rate Maximizing Sub-carrier Allocation is NP-hard.*

proof: The proof of Subset Sum $\leq_P$ Continuous Rate Maximizing Sub-carrier Allocation is line by line similar to Theorem 1, except the rate-power equations: the power for rate 0 is 0, the power of rate $\frac{R}{|W|} - 1$ is no less than $w_i - \frac{1}{|W|}$, the power of rate $\frac{R}{|W|}$ is no more than w_i, the power of rate $\frac{R+1}{|W|}$ is no less than $\sum_{i=1}^{m} w_i - 1$, and the rate-power relation between rate 0 and infinity is continuous and convex. This constructed rate-power equation can be plotted as Figure 3.

We claim that Subset Sum has a satisfying solution if and only if Continuous Rate Maximizing Sub-carrier Allocation has an assignment, which honors two users' power budgets V and $(\sum_{i=1}^{m} w_i - V)$ and has total rate no less than R. Suppose Subset Sum has a solution such that the sum of the subset S is exactly V. Then, we can allocate every corresponding sub-carrier in S to one user, and all the remaining sub-carriers to the other user. Moreover, every sub-carrier is loaded with rate exactly $\frac{R}{|W|}$. This assignment can be a satisfying solution to Continuous Rate Maximizing Sub-carrier Allocation. On the other hand, if Subset Sum is a NO instance, then no subset can give us a sum exactly V. Because every w_i is a natural number, the difference of sum between any subset and V must be no less than 1. In addition, the largest number of sub-carrier a user can be allocated is $|W| - 1$, and the highest power budget of these two user is at most $\sum_{i=1}^{m} w_i - 1$. Consequently, one of the two users has to load power on one of the sub-carrier assigned to him/her with value no more than $w_i - \frac{1}{|W|}$. On the other hand, the other user can load power on at most one of the sub-carrier assigned to him/her with value no more than $\sum_{i=1}^{m} w_i - 1$. Therefore, the total rate must be less than R, which implies a NO instance of Continuous Rate Maximizing Sub-carrier Allocation.

By replacing the continuous constructed rate-power equations in the proof of Theorem 5 with discrete ones, a line by line similar proof can give us the following conclusion:

Theorem 6 *Discrete Rate Maximizing Sub-carrier Allocation is NP-hard.*

4.2 In-approximability Proof

In the previous section we rigorously demonstrated a standard assumption that various versions of the sub-carrier problem are fundamentally hard. We next compute to what extend deterministic polynomial time algorithms can approximate the optimal solution in such scenarios. Gap-introducing technique[15] is adopted in the following proofs.

Theorem 7 *Achieving an approximation ratio $\alpha, \forall\alpha \geq 1$ for Continuous Downlink Sub-carrier Allocation is NP-hard.*

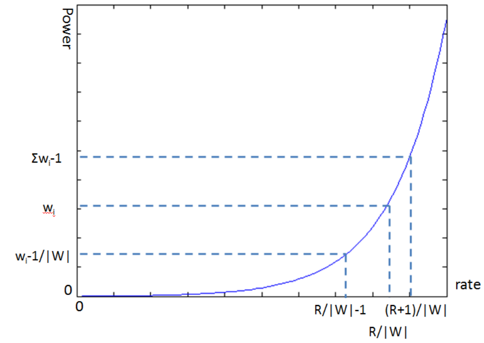

Figure 3. An example rate-power curve of a constructed sub-carrier in Continuous Rate Maximizing Sub-carrier Allocation.

proof: The proof of Subset Sum $\leq_P$ Continuous Downlink Sub-carrier Allocation is line by line similar as Theorem 3, except that the power for rate $w_i + \frac{1}{|W|}$ is no less than αP. The rate-power equations of constructed sub-carriers can be plotted in Figure 4.

We claim that Subset Sum has a satisfying solution if and only if an $\alpha-$approximation algorithm of Continuous Downlink Sub-carrier Allocation generates an solution with total power consumption at most αP. Suppose Subset Sum has a solution such that the sum of the subset S is exactly V, then we know the optimal solution for the corresponding Continuous Downlink Sub-carrier Allocation is no more than P. Consequently, an $\alpha-$approximation algorithm of Continuous Downlink Sub-carrier Allocation will give a solution that is at most αP.

On the other hand, if Subset Sum is a NO instance, then no subset can give us a sum exactly V. Because every w_i is a natural number, the difference of sum between any subset and V must be no less than 1. In addition, the largest number of sub-carrier a user can be allocated is $|W| - 1$. Consequently, one of the two users has to allocate at least one of the sub-carrier assigned to him/her with rate higher than $w_i + \frac{1}{|W|}$. Therefore the optimal maximal individual power consumption is higher than αP, and so is the total power consumption, which implies the solution provided by the $\alpha-$approximation algorithm must be higher than αP.

Similarly, by replacing the constructed rate-power equations in the proof of Theorem 7 with discrete ones similar to Figure 4, we can have the following theorem:

Theorem 8 *Achieving an approximation ratio $\alpha, \forall\alpha \geq 1$ for Discrete Downlink Sub-carrier Allocation is NP-hard.*

Using a similar technique as we prove Theorem 1 and 3, 1 and 2, we can have the following conclusions:

Theorem 9 *Achieving an approximation ratio $\alpha, \forall\alpha \geq 1$*

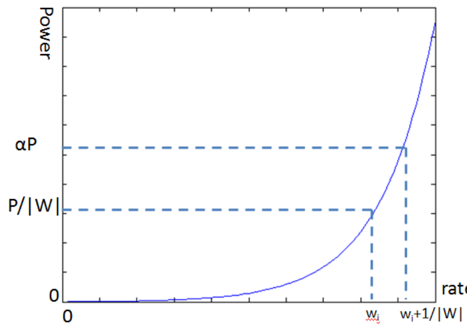

Figure 4. An example rate-power curve of a constructed sub-carrier in Continuous Uplink Sub-carrier Allocation.

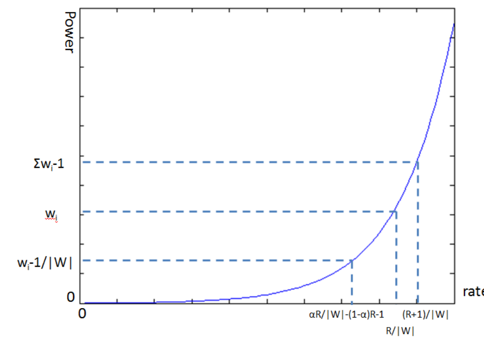

Figure 5. An example rate-power curve of a constructed sub-carrier in Continuous Rate Maximizing Sub-carrier Allocation.

for Continuous Uplink Sub-carrier Allocation and Discrete Uplink Sub-carrier Allocation are both NP-hard.

For the Continuous Rate Maximizing Sub-carrier Allocation, we have a slightly different conclusion:

Theorem 10 *Achieving an approximation ratio $\frac{m}{1+m}$ for Continuous Rate Maximizing Sub-carrier Allocation is NP-hard.*

proof: Consider the above reduction of Subset Sum to Continuous Rate Maximizing Sub-carrier Allocation with the following changes. The power for rate $w_i - \frac{1}{W}$ is no less than $\frac{\alpha R}{|W|} - (1-\alpha)R - 1$, and can be plotted in Figure 5.

We claim that Subset Sum has a solution if and only if an $\alpha-$approximation algorithm of Continuous Rate Maximizing Sub-carrier Allocation gives a solution with total rate at least αR. Suppose Subset Sum has a solution such that the sum of the subset S is exactly V, then we know the optimal solution for the corresponding Continuous Rate Maximizing Sub-carrier Allocation is no less than R. Consequently, an $\alpha-$approximation algorithm of Continuous Rate Maximizing Sub-carrier Allocation will give a solution that is no less than αR.

If Subset Sum is a NO instance, then no subset can give us a sum exactly V. Because every w_i is a natural number, the difference of sum between any subset and V must be no less than 1. In addition, the largest number of sub-carrier a user can be allocated is $|W| - 1$, and the highest power budget a user can have is $\sum w_i - 1$. Consequently, one of the two users has to allocate one of the sub-carrier assigned to him/her with power at most $w_i - \frac{1}{|W|}$. On the other hand, the other user can load at most one of the sub-carrier allocated to him/her with power no higher then $\sum w_i - 1$, thus the total rate these two users can have must be less than αR, and so is the output of the $\alpha-$approximation algorithm.

However, in order to make the reduction valid, we need $\frac{\alpha R}{|W|} - (1-\alpha)R - 1 > 0$, which implies $\alpha > \frac{|W|}{1+|W|}$. Because $m = |W|$, thus completes our proof.

Again, using a line by line similar proof with discrete version of Figure 5, we can have the following theorem:

Theorem 11 *Achieving an approximation ratio $\frac{m}{1+m}$ for Discrete Rate Maximizing Sub-carrier Allocation is NP-hard.*

5. Simulation and Discussion

Although we have proved the in-approximability of various sub-carrier allocation problems, we also notice that a local search based algorithm, which is referred to *k-interchange* and is listed in 1, performs close to optimal in all of our simulation instances.

5.1 k-interchange Algorithm

Algorithm 1 k-Interchange Algorithm for Uplink

Start with t = 0, and E = $\{(u_i, s_j) | i = 1, \ldots, m; j = 1, \ldots, n\}$.
Randomly assign all sub-carriers to users under the feasibility constraint.
We denote this assignment set as S^0.
while $\exists V \subset E$ under feasibility constraint, such that $|V| = m$, $|V - S^t| = |S^t - V| \leq k$, and the objective can be improved by more than the threshold factor ϵ **do**
    Let $S^t = V$
    $t = t + 1$
end while

The k-interchange algorithm works in the following fashion. It starts with a random *feasible* solution, wherein a solution comprises feasible sub-carrier allocation to users, followed by a feasible rate loading on assigned sub-carriers by the respective users. It then explores solutions in the neighborhood of the current one, by randomly permuting up to k sub-carriers across users and identifying an allocation which can improve existing solution by an amount more than some *a priori* chosen factor ϵ. If no such solution is found, the algorithm terminates, else it proceeds to explore the k-neighborhood of the new solution.

k-interchange algorithm is consisted of two parts. In the first part, k-interchange needs to assign all sub-carriers to all users under the feasibility constraints stated in previous section. In the second part, after every user being assigned a set of sub-carriers, he/she needs to load power onto each of these sub-carriers to satisfy his/her date rate demands.

5.2 Simulation Environments

In our simulation, we use 1-, 2-, and 3-interchange algorithm, i.e. $k = \{1, 2, 3\}$, and compare their solutions with optimal ones, which are computed by exhaustive search. There are 2 to 5 users, and all of their rate requests are 10. Up to 10 sub-carriers downlinks are available, and the number of sub-carriers is strictly larger than the number of users in every instance. (Note that computation of the optimal solution is computationally very expensive and hence we are forced to limit scenarios to small problems.) Every sub-carrier has the same bandwidth, and we assume the single-sided noise Power Spectral Density for every sub-carrier and every user is 1. The signal propagation environment utilized a simplified path loss model combined with log-normal shadowing [4]. The path loss statistics were assumed to be independent across users with the path loss factor chosen uniformly in the range $[-10\text{dB}, -20\text{dB}]$ and the exponent in the range $[2, 3]$. The log-normal shadowing model is characterized by a random variable ψ_{dB} with normal distribution $N(0dB, 3dB)$. M-QAM is adopted, and the required power for supporting c bits/symbol at a given BER is provided in Wong's paper [17]. In this paper, we assume the BER for every user is 10^{-4}. In addition, the value of ϵ used for k-interchange algorithm is 0.01.

5.3 Simulation Results

In Figure 6, 7, and 8, we plot the required bit SNR (in dB) for the optimal and 2-interchange algorithm solutions by varying user number, sub-carrier number and the variance of log-normal model. The parameters used in these figures are described in the corresponding captions, and the value plotted are averaged over 100 iterations under different settings. Note that, although we only plot the results for the settings specified in the captions, the results for all the other parameter combinations are very similar to what we present here.

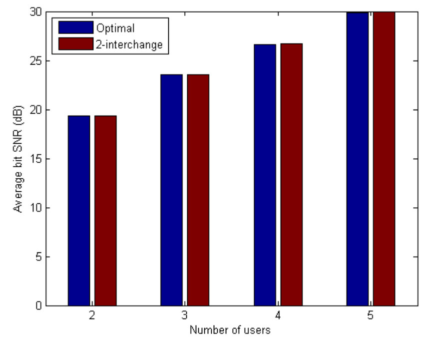

Figure 6. Required average bit SNR (dB) of optimal and 2-interchange solutions. Parameters: $m = 10$, $n = \{2, 3, 4, 5\}$.

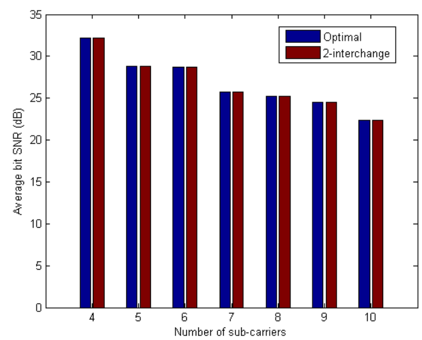

Figure 7. Required average bit SNR (dB) of optimal and 2-interchange solutions. Parameters: $m = \{4, 5, 6, 7, 8, 9, 10\}$, $n = 3$.

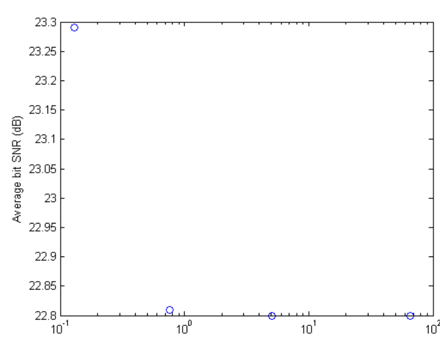

Figure 8. Required average bit SNR (dB) vs. average computing time per iteration for various k-interchange and optimal solution. Parameters: $m = 10$, $n = 3$.

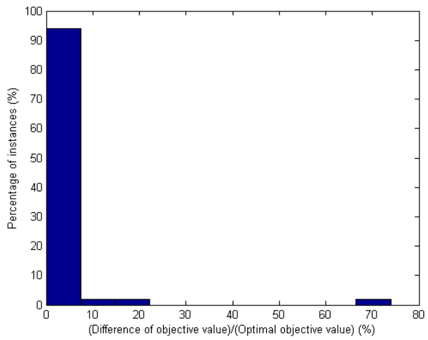

Figure 9. Histogram of (Difference of objective value between 1-interchange and optima)/(Optimal objective). Parameters: $m = 10$**,** $n = 3$**.**

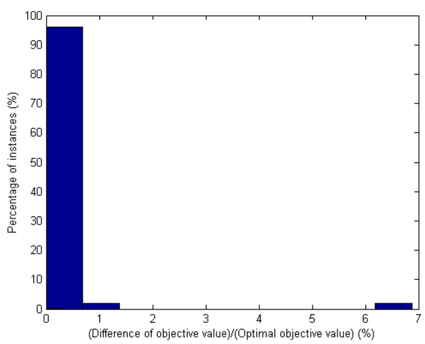

Figure 10. Histogram of (Difference of objective value between 2-interchange and optima)/(Optimal objective). Parameters: $m = 10$**,** $n = 3$**.**

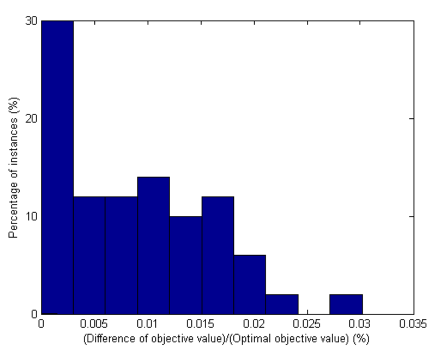

Figure 11. Histogram of (Difference of objective value between 3-interchange and optima)/(Optimal objective). Parameters: $m = 10$**,** $n = 3$**.**

As plotted Figure 6, when user number increases, the average bit SNR increases. Because every sub-carrier must be allocated exclusively, every sub-carrier needs to carry higher rate in average when user number increases, thus raising the required average bit SNR. In Figure 7, when channel number increases, the average bit SNR decreases. Since the rate that each sub-carrier needs to provide reduces as the sub-carrier number increases, it can be expected that the average bit SNR decreases. Among these figures, there is one thing in common: the solutions generated by 2-interchange algorithm has close to optimal quality. Although the performance of 2-interchange is promising in our simulations, a natural question we would like to ask is: *how sensitive the performance degradation is when the neighborhood definition becomes different, i.e. performance variation vs. different values of* k*?* In order to answer this question, we plot the average per iteration computing time of $1-, 2-, 3-$-interchange and their average bit SNR in Figure 8. For comparison purpose, we also plot the optimal average bit SNR in the same figure.

The circles from left to right represent $1-$, $2-$, $3-$interchange, and optimal solution, respectively. As can be observed from Figure 8, the highest gain is between $1-$ and $2-$interchange solution, and the average difference is 0.48 dB. Although 3-interchange is still better than 2-interchange in average, the difference is merely 0.01 dB, and both their solutions are within 0.02 dB from the optimal solutions in average. However, the computing time for various k value are significantly different. While all 3 k-interchange algorithms terminate within 5.1 seconds in average for one iteration, the optimal solution takes 65.7 seconds. Even for the case of 5 users and 10 sub-carriers, all k-interchange can finish within 28 seconds in average, but the optimal solution takes around 6,250 seconds. To further understand the solution quality for each k value, we plot the histogram of power difference from optimal solutions in Figure 9, 10, and 11.

As we can see from Figure 9, although over 90% of instances using 1-interchange are within 10% difference from the optima, this gap can go up to around 70% in the worst case scenario. On the other hand, in Figure 10, the worst case of using 2-interchange algorithm is still within 7% gap from optima, and over 95% of instances, this gap is within 1%. In addition, as depicted in Figure 11, the gap between 3-interchange solutions and optima is no more than 0.03%.

In sum, although using a large neighborhood definition, i.e. value of k increases, can give us better average solution quality, but the marginal improvement diminishes rapidly. On the contrary, the computing time for different k increases in the order of sub-carrier number.

5.4 When k-interchange Fails

Although the solution quality of k-interchange looks promising in our simulations, we would like to demonstrate an example when the k-interchange algorithm may perform arbitrarily bad.

Consider the following 3-users (user A, B, and C), 4-channels (channel I, II, III, and IV) situations for 2-interchange algorithm. For convenience, we use $\gamma_{i,j}$ to represent the channel gain for user i with respect to sub-carrier j.

- $\gamma_{A,I} = 10^{-3}$, $\gamma_{A,III} = 1$, and $\gamma_{A,II} = \gamma_{A,IV} = \infty$.
- $\gamma_{B,II} = 10^{-3}$, $\gamma_{B,I} = 1$, and $\gamma_{B,III} = \gamma_{B,IV} = \infty$.
- $\gamma_{C,III} = 10^{-3}$, $\gamma_{C,II} = 1$, and $\gamma_{C,I} = \gamma_{C,IV} = \infty$.
- The rate requests of user A, B, and C are all 1.
- $power = \frac{1}{\gamma} \times (2^{rate} - 1)$

One possible output of 2-interchange algorithm can be $\{d_{A,I} = 1, d_{B,II} = 1, d_{C,III} = 1, d_{C,IV} = 1\}$. However, the optimal solution for this instance is $\{d_{A,III} = 1, d_{B,I} = 1, d_{C,II} = 1, d_{C,IV} = 1\}$. Because every eligible 2-interchange movement increases the total power consumption to infinity, thus it is impossible to transform the solution of 2-interchange to the optimal one.

However, if we are allowed to reallocate 3 sub-carriers at once, i.e. using $3-$interchange algorithm instead, then we can greatly reduce the total power consumption (from 3000 to 3), as well as maximal individual power consumption (from 1000 to 1). In addition, this gap can be arbitrarily bad as we decrease the value of $\gamma_{A,I}, \gamma_{B,II}, \gamma_{C,III}$. Therefore, 2-interchange does not have a bound. Similarly, we can construct a worst case example for any $k-$interchange algorithm, that makes the solution quality arbitrarily bad.

In sum, when channel gains have very high variability, which prevents k-interchange from making small moves, it is possible that the solution quality of the proposed algorithm is bad.

6. Conclusion

In this paper, we study a multi-user/multi-subcarrier allocation problem, which can be applied to OFDM systems. We formally prove that the sub-carrier resource allocation problem is NP-hard for both power minimization as well as rate maximization in both uplink and downlink scenarios. In addition, we also demonstrate a set of in-approximability results which indicate that deterministic polynomial time algorithms cannot hope to always provide good performance for sub-carrier allocation, especially when power minimization is concerned. This points to the direction of randomized algorithms as an alternative promising approach, which we hope to address in future work.

Although the sub-carrier allocation problems with power related objective have been proved in-approximable, the proposed algorithm, $k-$interchange, performs close to optimal in all our simulations. In addition, we also identify the worst case scenario, such that using $k-$interchange may lead to arbitrarily bad solution quality.

As an extension, we plan to do extensive numerical comparison between our algorithm and other existing efforts by simulations. It is also of interest to investigate the effect of an adaptive threshold value, and strategies to reduce running time of $k-$interchange algorithm.

7. Acknowledgment

The work described here is supported in part by NSF through grants CNS-0347621, CNS-0627028, CCF-0430061, CNS-0325875, NASA through an AIST grant. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF, NASA, or USC Viterbi School of Engineering.

References

[1] T. Alen, A. Madhukumar, and F. Chin. Capacity enhancement of a multi-user ofdm system using dynamic frequency allocation. *IEEE*, 2003.

[2] T. Cover and J. Thomas. Elements of information theory. 1991.

[3] A. Czylwik. Adaptive ofdm for wideband eadio channels. *IEEE Globecom'96, London, UK*, November 1996.

[4] A. Goldsmith. *Wireless Communications*. Cambridge University Press, 2005.

[5] IEEE. 802.16: Worldwid interoperability for microwave access (wimax). December 2001.

[6] IEEE. 802.11g: Lan/man standards committee. June 2003.

[7] I. Kim, H. Lee, B. Kim, and Y. Lee. On the use of linear programming for dynamic subchannel and bit allocation in multiuser ofdm. *IEEE*, 2001.

[8] D. Kivanc, G. Li, and H. Liu. Computationally efficient bandwidth allocation and power control for ofdma. *IEEE*, 2003.

[9] J. Kleinberg and E. Tardos. *Algorithm Design*. Addison-Wesley, 2006.

[10] M.Bohge, J. Gross, and A. Wolisz. A new optimization model for dynamic power and sub-carrier allocations in packet-centric ofdma cells. *Proc. Of 11th International OFDM-Workshop*, August 2006.

[11] W. Rhee and J. Cioffi. Increase in capacity of multiuser ofdm system using dynamic subsubcarrier allocation. *Proceedings of VTC*, 2000.

[12] H. Rohling and R. Grunheid. Performance comparison of different multiple access schemes for the downlink of an ofdm communication system. *Proc. IEEE VTC'97, Phoenix, AZ*.

[13] H. Rohling and R. Grunheid. Performance of an odfm-tdma mobile communication system. *Proc. IEEE VTC'96, Atlanta, GA*, 1996.

[14] Z. Shen, J. Andrews, and B. Evans. Adaptive resource allocation in multiuser ofdm systems with proportional rate constraints. *IEEE Transactions on Wireless Communications*, 4, November 2005.

[15] V. Vazirani. Approximation algorithms. 2003.

[16] M. Wahlqvist, H. Olofsson, M. Ericson, C. Ostberg, and R. Larsson. Capacity comparison of an ofdm based multiple access system using different dynamic resouce allocation. *IEEE*, 1997.

[17] C. Wong and R. Cheng. Multiuser ofdm with adaptive subcarrier, bit, power allocation. *IEEE JSAC*, 17(10), October 1999.