Performance Bounds of Asynchronous Circuits: Lemmas and Theorems

Mehrdad Najibi and Peter A. Beerel

Ming Hsieh Department of Electrical Engineering
University of Southern California
Los Angeles, California 90089
najibiko@usc.edu, pabeerel@usc.edu

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1 Introduction

This document provides the detailed proofs for lemmas and theorems of [2]. For the definitions and more explanations refer to the paper.

2 Lemmas, Theorem, and Proofs

Lemma 1. *Let $\hat{p}_1$ and $\hat{p}_2$ be two places in a segment of a live safe and reversible unique choice Petri net with all cycles initially marked. If $\hat{p}_1$ is not on the boundaries of the segment, then $l(\hat{p}_1) \neq l(\hat{p}_2)$.*

Proof. (By contradiction) If $l(\hat{p}_1) = l(\hat{p}_2) = p$, there must exist a path, ρ, between $\hat{p}_1$ and $\hat{p}_2$ in the segment. Otherwise $\hat{p}_1$ and $\hat{p}_2$ would be concurrent which would imply the existence of a marking in the original Petri net with multiple tokens in p, contradicting the fact that the Petri net is safe. This path corresponds to a cycle, c, in the original Petri net containing p which is not initially marked, contradicting our initial assumption. □

Lemma 2. *For any two arbitrary transitions in any segment of a live safe and reversible unique choice Petri net with all cycles marked initially, $l(\hat{t}_1) \neq l(\hat{t}_2)$.*

Proof. (By contradiction) All the transitions in any segment have all their input places in the segment. If $l(\hat{t}_1) = l(\hat{t}_2)$ then either there exists two input places where $l(\hat{p}_1) = l(\hat{p}_2)$ or there exists one input place in which both transitions $\hat{t}_1$ and $\hat{t}_2$ are in its post-set. The latter suggests that the segment exhibits choice which is not possible by the definition of a segment. The former is not possible according to Lemma 1. $\square$

Lemma 3. *Let $s_i = \langle \tilde{P}_i, \tilde{T}_i, \tilde{F}_i \rangle$ be a segment of an unfolded execution of a live safe and reversible unique choice Petri net. $\pi(s_i)$ is a live safe marked graph.*

Proof. $\pi(s_i)$ is a marked graph because by definition of a marked graph all unfolded places have at most one output and one input transition and by Lemma 1 and Lemma 2 no two internal unfolded places/transitions in a segment map to the same original place/transition. $\pi(s_i)$ is live since s_i is acyclic so by marking all places on its input cutline, all the paths in s_i are marked by exactly one token. Therefore each cycle, $c \in \sum(s_i)$, is marked at least by one token, a sufficient condition to guarantee $\pi(s_i)$ is live. Moreover, from each input boundary place p there must exist a path to its next occurence p'. Otherwise, otherwise p and p' are concurrent in the unfolded execution and a reachable marking exists in which $l(p)$ is marked with more than one token, contradicting the fact that the original Petri net P is safe. Thus, every initially marked place in $\pi(s_i)$ is part of a cycle that contains exactly one token, a sufficient condition to guarantee that $\pi(s_i)$ is safe. $\square$

Lemma 4. *Let s_i and s_j be two segments of an unfolded execution of a live safe and reversible unique choice Petri net that has every cycle marked with at least one token. Let s^*_{ij} be the super segment obtained by $s_i \times s_j$. $\pi(s^*_i)$ is a live safe marked graph.*

Proof. $\pi(s^*_{ij})$ is a marked graph because cross operator will not introduce any choice or merge by definition. The cross operator generates acyclic super segments, otherwise there must exist a cycle in the original Petri net that has no token, a contraction to our initial assumption. Thus, using the same argument as in the proof of Lemma 3, we know $\pi(s^*_{ij})$ is live. Moreover, if ether s_i or s_j has a path from a place p to its next occurence $p', l(p') = l(p)$, a path would exists between these two places in s^*_{ij}. If neither s_i nor s_j has a path then $p = p'$, this place will not exist in s^*_{ij}. Thus, every place p on the input cutline of s^*_{ij} has a path to its next occurrence $p', l(p') = l(p)$. This means that every initially marked place in $\pi(s^*_{ij})$ is part of a cycle that contains exactly one token, a sufficient condition to guarantee that a marked graph is safe [1]. $\square$

Lemma 5. *Let $U^*[i : j]$ be a sequence of segments obtained from $U[i : j]$ by elevating s_k to s^*_k. For any path, ρ, between two unfolded places or transition, x_i and x_j in $U[i : j]$, there exists a path, ρ^*, between them in $U^*[i : j]$ such that $\rho \subseteq \rho^*$.*

Proof. Let ρ cross s_k boundaries at places p_s and p_e.

Case 1: If there exists a path between p_s and p_e such that $p_s \neq p_e$, then there must exist a path from $p_s^* = p_s$ to $p_e^* = p_e$ in s_k^* because the cross operator only add arcs between unfolded transition or places. Moreover, ρ^* essentially passes along the same path and $p_s^* = p_s$ to $p_e^* = p_e$. The cross operator might insert additional transitions on this path to convert choices/merges constructs to fork/join, but by dropping these additional transitions ρ_{k-1}^* can be reproduced.

Case 2: If there is no path between p_s and p_e then we must have $p_s = p_e$. There are two sub-cases: *Case 2-1*: $p_s^* = p_e^*$. In this case, $p_s^* = p_s$ and ρ^* is the same as ρ. *Case 2-2*: $p_s^* \neq p_e^*$. In this case, $p_s^* = p_s$ and p_e^* is the next occurrence of the p_s^* in $U^*[i:j]$. Since $\pi(s_k^*)$ is live and safe, every initially marked place in $\pi(s_k^*)$ must exist in a cycle containing exactly one token [1]. Thus, there must exist a path between any place on the input cutline of s_k^* to the next occurrence of the same place on the output cutline of s_k^*. In this case, ρ^* passes along the path from p_s^* to p_e^* and ρ can be reproduced by dropping all transitions and places along this path. □

Lemma 6. *Let U^* be a sequence of segments obtained from U by elevating s_j to s_j^*. $\bar{\gamma}_U(t) \leq \bar{\gamma}_{U_j^*}(t)$.*

Proof. Let the globally critical path in U be ρ. By Lemma 5 this path is a subset of a path ρ^* contained in U^*. Moreover, as all delays in U^* are non-negative, we know the critical path in U^* is at least as long as that of U. □

As introduced in Section **??**, we assume the Petri net has m segment types $\bar{S} = \{s_1, ..., s_m\}$ corresponding to mode set $\bar{\mu} = \{\mu_1, ..., \mu_m\}$. We consider an unfolded execution of length N with mode sequence $U = \langle U_0, ..., U_N \rangle$ where $U_j \in \bar{\mu}$. Our goal is to bound the length of the longest path from the first to the last instance of the transition of interest t in this sequence, i.e., to bound $\gamma_U^{(0)}(t, t, N)$.

We define increasingly larger super segments $\bar{S}^* = \{s_1^*, ..., s_m^*\}$, where $s_1^* = s_1$, and $\forall i \neq 1, s_i^* = s_{i-1}^* \times s_i$. We let τ_i^* be the cycle time for the marked graph associated with super segment s_i^*.

Lemma 7. $\forall i > 1, \tau_i^* \geq \tau_{i-1}^*$.

Proof. Based on Lemma 6 since $s_{i-1}^* \subseteq s_i^*$ we can replace every s_{i-1}^* in U_{i-1}^* with s_i^* to get U_i^*. □

Theorem 1. *For any arbitrary unfolded execution of a live safe and reversible unique choice Petri net, U, we have $\bar{\gamma}_U(t) \leq \tau_m^*$.*

Proof. By Lemma 6 since $\forall s \in \mu, s \subseteq s^*_m$ we can replace each segment in U to get U^*_m which completes the proof. ☐

Theorem 1 bounds the cycle time of any arbitrary unfolded execution of the Petri net with conditionals from above without any assumption about the order of the segments (modes) or their frequencies. This result was intuitively accepted by industry and the bound is used for slack matching of asynchronous circuits with conditional behavior by treating them as unconditional.

The bound of Theorem 1 is not optimal. When applied to slack matching, this conservative bound results in extra slack matching buffers. In the following we will try to obtain tighter upper bounds for the cycle time considering more assumptions about the mode switching behavior of the Petri net.

To do this, as introduced in Section **??**, we consider an unfolded execution $U = \langle U_0, ..., U_N \rangle$ and start by proving a useful lemma about the time separation of events within subsequences of segments that have the same type.

Lemma 8. $\forall \rho$ *Lets define the span of each path,* ρ*, denoted by* $||\rho||$ *is the number of cutline intersecting with that path.*

$$\forall \, \hat{p}_1, \hat{p}_2 \in \hat{P}, l(\hat{p}_1) \neq l(\hat{p}_2) \Rightarrow ||\rho|| < |M_0|$$

Proof. Lets use $\rho(i)$ notation to denote the place where cutline c_i and path ρ intersect. Lets assume that there exists a place-simple path, ρ', with $k = ||\rho'|| > |M_0|$. Let $\{c_1, c_2, ..., c_k\}$ be the set of cutlines intersecting with ρ' and $\{\hat{p}_1, \hat{p}_2, ..., \hat{p}_k\}$ are places on the cutlines which intersect with the path, $\hat{p}_i = \rho'(c_i)$. Since ρ' is a place-simple cycle we should have $l(\hat{p}_1) \neq l(\hat{p}_2) \neq ... \neq l(\hat{p}_k)$ and all since these places are on the cutlines $\{l(\hat{p}_1), l(\hat{p}_2), ..., l(\hat{p}_k)\} \subset M_0$ which is a contradiction since $k > |M_0|$ by assumption. ☐

Lemma 9. *Cycle extraction as defined in section* **??** *preserve the following property:*

$$D(\rho^*) = \sum_i (\rho^*_i)$$

Proof. ☐

Lemma 10. $D(\rho^*_i) \leq |E^i| \tau(s^*_{Max}(E^i))$

Proof. ☐

Theorem 2. $\lim_{N\to\infty} \gamma_U^{(0)}(t,t,N) \leq \tau_B(U)$*, where*

$$\tau_B(U) = \sum_i |E^i| \tau(s^*_{Max}(E^i)).$$

Proof. □

Theorem 3. *For an arbitrary mode assignment to* $\{E^i\}$, $1 \leq i \leq \eta$*, let* $\tau_B(\{E^i\}) = \sum_i |E^i|\tau(s^*_{Max}(E^i))$ *then* $\tau_B(\{E^i\}) \leq \tau^*_B = \tau(\{\hat{E}^i\})$

$$\hat{E}^i = \langle s^*_i, U^i_{MIN}(|M_0|)\rangle.$$

Proof. **Exchange Argument:** We prove that any arbitrary $\{E^i\}$ can be converted into $\{\hat{E}^i\}$ trough a series of mutations. Initially, $\{\tilde{E}^i\} = \{E^i\}$ and lets assume at step j of conversion $\tilde{E}^{1:j} = \hat{E}^{1:j}$, and $\tau(\{E^i\}) \leq \tau(\{\tilde{E}^i\})$ we show that after step $j+1$ we will have $\tilde{E}^{1:j+1} = \hat{E}^{1:j+1}$ and

$$\tau(\{E^i\}) \leq \tau(\{\tilde{E}^i\}).$$

At step $j+1$ some $\tilde{E}^k$, $k \geq j+1$ has the largest mode. Lets swap $\tilde{E}^k$ and $\tilde{E}^{j+1}$. We then swap the largest segment in $\tilde{E}^{j+1}$ with its first segment. Clearly these two exchanges wont change $\tau(\{\tilde{E}^i\})$. We then exchange each segment in $\tilde{E}^{j+1}$, from second to the last segment, with the smallest segment in $\tilde{E}^{i\geq j+1}$. Let assume that the smallest segment falls into $\tilde{E}^k$ if the next smallest segment is exchanged for a segment of $\tilde{E}^{j+1}$ which is larger than $s^*_{Max}(\tilde{E}^k)$, $\tau(\{\tilde{E}^i\})$ increases as $\tilde{E}^k$ has to be elevated to a larger segment otherwise there would be no change in elevation and $\tau(\{\tilde{E}^i\})$ remains constant. The elevation of $\tilde{E}^{j+1}$ is governed by its first segment which remains unchanged.

Finally, we increase the length of $\tilde{E}^{j+1}$ one at a time until we reach to $|M_0|$, by removing next smallest segment from $\tilde{E}^k$, $k \geq j+1$, and appending it to $\tilde{E}^{j+1}$. On this exchange $\tau(\{\tilde{E}^i\})$ will be increased by $\Delta = \tau(s^*_{MAX}(\tilde{E}^{j+1})) - \tau(s^*_{MAX}(\tilde{E}^k))$ and we know that $\Delta \geq 0$ because $\tilde{E}^{j+1}$ contains the maximum segment.

By the sequence of exchanges applied, we know have $\tilde{E}^{j+1} = \hat{E}^{j+1}$ and as shown by each exchange $\tau(\{\tilde{E}^i\})$ either remains constant or increases and therefor:

$$\tau(\{E^i\}) \leq \tau(\{\tilde{E}^i\})$$

which completes the proof. □

Theorem 4. *For any arbitrary long sequence U with arbitrary mode order,* $\tau_U \leq \sum_{j=1}^{m} \hat{f}_j \tau(s_j^*)$ *where*

$$\begin{cases} \hat{f}_m = Min(f_m|M_0|, 1) \\ \hat{f}_j = (1 - Min(\sum_{k>j} \hat{f}_k, 1)) f_j |M_0|, \;\; j < m \end{cases}$$

Proof. **by Induction** By theorem 3 we know that $\tau_U \leq \sum_i |E^i| \tau(s^*_{Max}(E^i))$. Lets assume that the sequence has N segments. For the base case, $\{\hat{E}^i\}, i \leq f_m N$ are affected by mode s_m^* which result in total $N_m = min(|M_0| f_m N, N)$ mo.de elevations to mode m leaving $(N - N_m)$ un-elevated modes. $\hat{f}_m$ can be calculated as:

$$\hat{f}_m = \lim_{N \to \infty} N_m / N = Min(f_m |M_0|, 1)$$

After elevating for modes $m, m-1, ..., j+1$ we have $N_{m:j+1} = Min(\sum_{k>j} f_k |M_0| N, N)$ segments already elevated and $N - N_{m:j+1}$ segments left un-elevated. On jth elevation we elevate $(N - N_{m:j+1}) f_j |M_0|$ segments to S^*_{j+1}. We will have

$$\hat{f}_j = (1 - \lim_{N \to \infty} \frac{N_{m:j+1}}{N}) f_j |M_0|$$

$$= (1 - Min(\sum_{k>j} \hat{f}_k, 1)) f_j |M_0|$$

□

References

[1] T. Murata. Petri nets: Properties, analysis and applications. *Proceedings of the IEEE*, 77(4):541–580, April 1989.

[2] Mehrdad Najibi and Peter A. Beerel. Performance bounds of asynchronous circuits with mode-based conditional behavior. In *submitted to IEEE International Symposium on Asynchronous Circuits and Systems*, 2012.