Static LQG team with convex cost function

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Project Report

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I. Introduction

In team decision problems, there are multiple decision-makers/team members, each controlling different actions or decision variables and having access to different information. However, there is a single common goal or cost function for all members. A team decision problem is referred to as static if the information available to each member is independent of the actions of other members. For static team problems with Gaussian random variables, linear observations and quadratic, strictly convex cost functions, it has been shown that there exists a unique optimal decision strategy and that the optimal strategy is a linear function of the members' information [1], [2], [3]. In this report, instead of assuming that the quadratic cost function is strictly convex, we assume that it is only convex. Hence, the results of [1], [2], [3] cannot be directly applied here.

We show that for static team problems with Gaussian random variables, linear observations and quadratic, convex cost functions, linear decision strategies are optimal if a certain linear system of equations has a solution.

II. Notation

Uppercase letters denote random variables/vectors and their corresponding realizations are represented by lowercase letters. Uppercase letters are also used to denote matrices. Almost sure equality between random variables X and Y is denoted by $X \stackrel{a.s.}{=} Y$. $\mathbb{E}[\cdot]$ denotes the expectation of a random variable. For collection of functions $\boldsymbol{g}$, $\mathbb{E}^{\boldsymbol{g}}[\cdot]$ denotes that the expectation depends on the choice of functions in $\boldsymbol{g}$. When random variable X is normally distributed with mean μ and variance Σ, it is shown as $X \sim \mathcal{N}(\mu, \Sigma)$.

For a sequence of column vectors $X, Y, Z, ...$, the notation $vec(X, Y, Z, ...)$ denotes vector $[X^\intercal, Y^\intercal, Z^\intercal, ...]^\intercal$. Furthermore, the vector $vec(X_1, X_2, ..., X_t)$ is denoted by $X_{1:t}$. The transpose, Moore-Penrose pseudo-inverse, and trace of matrix A are denoted by $A^\intercal$, $A^\dagger$, and $\mathbf{tr}(A)$, respectively. I_n denotes a $n \times n$ identity matrix. We omit the subscripts when dimensions can be inferred from context.

A matrix can be partitioned as $A = \begin{bmatrix} A_{r,1}^\intercal & A_{r,2}^\intercal & \dots & A_{r,n}^\intercal \end{bmatrix}^\intercal$ and $A = \begin{bmatrix} A_{c,1} & A_{c,2} & \dots & A_{c,n} \end{bmatrix}$ where the dimensions of $A_{r,i}$ and $A_{c,i}$ are inferred from the context. Furthermore, the vectorization of a $m \times n$

matrix A is denoted by A_v and is formed by stacking all the columns of A into a column vector, that is, if $A = \begin{bmatrix} A_{c,1} & A_{c,2} & \dots & A_{c,n} \end{bmatrix}$ where each $A_{c,j}$ is an $m \times 1$ matrix, then,

$$A_v = \begin{bmatrix} A_{c,1} \\ A_{c,2} \\ \vdots \\ A_{c,n} \end{bmatrix}.$$

If A is an $m \times n$ matrix and B is a $p \times q$ matrix, then the Kronecker product $A \otimes B$ is a $mp \times nq$ block matrix formed as follows,

$$A \otimes B = \begin{bmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,n}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1}B & a_{m,2}B & \cdots & a_{m,n}B \end{bmatrix}$$

where $a_{i,j}$ is the entry in the i-th row and j-th column of matrix A. For two matrices A and B with the same dimension $m \times n$, the Hadamard product $A \odot B$ is a matrix of the same dimension and is defined as,

$$A \odot B = \begin{bmatrix} a_{1,1}b_{1,1} & a_{1,2}b_{1,2} & \cdots & a_{1,n}b_{1,n} \\ a_{2,1}b_{2,1} & a_{2,2}b_{2,2} & \cdots & a_{2,n}b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1}b_{m,1} & a_{m,2}b_{m,2} & \cdots & a_{m,n}b_{m,n} \end{bmatrix}$$

where $a_{i,j}$ and $b_{i,j}$ are the entry in the i-th row and j-th column of matrices A and B, respectively.

If set $\mathcal{A}$ is subset of set $\mathcal{B}$, it is denoted by $\mathcal{A} \subset \mathcal{B}$.

III. TEAM MODEL

Consider a team composed of n members denoted by $\mathcal{M} = \{1, 2, ..., n\}$. Let $\Xi \in \mathbb{R}^{d_\xi}$ be a random vector that represents all the uncertainties of the external world which are not controlled by any of the team members. Ξ is a Gaussian random vector: $\Xi \sim \mathcal{N}(0, \Sigma)$, where Σ is a positive definite matrix. The probability distribution of Ξ is assumed to be known to all members. The information available to member i is denoted by $Z^i \in \mathbb{R}^{d_{z^i}}$ which is a known linear function of Ξ:

$$Z^i = H^i \Xi \qquad \forall i \in \mathcal{M} \tag{1}$$

where H^i is a $d_{z^i} \times d_\xi$ matrix and is known to all members. Clearly, $Z^i \sim \mathcal{N}(0, H^i \Sigma H^{i\intercal})$. We assume that $H^i \Sigma H^{i\intercal}$ is positive definite.

Member i chooses control action $U^i \in \mathbb{R}^{d_{u^i}}$ as a function of Z^i, that is, $U^i = \gamma^i(Z^i)$ where γ^i is the control strategy of member i. We define the class of admissible control strategies for member i, denoted by Γ^i, as the set of all Borel measurable functions $\gamma^i : \mathbb{R}^{d_{z^i}} \to \mathbb{R}^{d_{u^i}}$ which have finite second moments, that is, $\mathbb{E}[\gamma^i(Z^i)^\intercal \gamma^i(Z^i)] < \infty$. The collection $\boldsymbol{\gamma} = (\gamma^1, \gamma^2, \dots, \gamma^n)$ where $\boldsymbol{\gamma} \in \Gamma = \Gamma^1 \times \Gamma^2 \times \dots \times \Gamma^n$

is called the control strategy of the team (or team strategy). The cost function $C(\Xi, U)$ is a quadratic function of Ξ and $U = \text{vec}(U^1, U^2, \ldots, U^n)$:

$$C(\Xi, U) = (M\Xi + NU)^\intercal(M\Xi + NU) = U^\intercal RU + 2U^\intercal S\Xi + \Xi^\intercal Q\Xi \tag{2}$$

where $R = N^\intercal N$, $S = N^\intercal M$, and $Q = M^\intercal M$. The dimensions of N and M are $d_s \times \sum_{i=1}^n d_{u^i}$ and $d_s \times d_\xi$ respectively. The performance of the control strategy $\boldsymbol{\gamma}$ is measured by the expected cost

$$\mathcal{J}(\boldsymbol{\gamma}) = \mathbb{E}^{\boldsymbol{\gamma}}\left[C(\Xi, U)\right]. \tag{3}$$

The optimization problem we consider is defined as follows.

Problem 1. *For the static team problem described above, find team control strategy $\boldsymbol{\gamma}$ that minimizes the expected cost given by* (3).

If R is positive definite, it has been shown that there exists a unique optimal decision strategy and that the optimal strategy is a linear function of the members' information [1], [2], [3]. In this report, we do not assume that R is positive definite. Hence, the results of [1], [2], [3] cannot be applied directly.

Remark 1. *The cost function is defined in terms of Ξ and the vector of all decisions, $U = \text{vec}(U^1, U^2, \ldots, U^n)$. For the sake of convenience, we will at times write the cost $C(\Xi, U)$ as $C(\Xi, U^1, \ldots, U^n)$. When the meaning is clear from the context, we may also rearrange the decisions in $C(\Xi, U^1, \ldots, U^n)$.*

IV. Optimal Strategies

For the team problem formulated above, suppose a team strategy $\boldsymbol{\gamma} \in \Gamma$ is such that

- $\mathcal{J}(\boldsymbol{\gamma})$ is finite,
- The partial derivatives in (4) are well-defined for all $i \in \mathcal{M}$,
- For each $i \in \mathcal{M}$ and for all realizations z^i of Z^i, define

$$F_{z^i}(u^i) := \mathbb{E}\left[C\big(\Xi, \{\gamma^j(Z^j)\}_{j\in\mathcal{M}\setminus\{i\}}, u^i\big)|Z^i = z^i\right].$$

$F_{z^i}(u^i)$ is the conditional expected value of the cost given $Z^i = z^i$ if all members $j \neq i$ use strategy γ^j and member i takes the action u^i. Suppose $\boldsymbol{\gamma}$ satisfies the following equations:

$$\nabla_{u^i} F_{z^i}(u^i)\Big|_{u^i=\gamma^i(z^i)} = 0, \;\; i \in \mathcal{M}. \tag{4}$$

A team strategy satisfying the above three conditions is called *stationary* [3].

Sufficient Conditions for Optimality of Stationary Strategies: [Theorem 2.6.5 [3]] Consider a static team problem satisfying the following conditions:

1) The cost function $C(\xi, u)$ is convex and continuously differentiable in u for all realizations ξ of Ξ,
2) $\mathcal{J}(\boldsymbol{\gamma})$ is bounded from below for all $\boldsymbol{\gamma} \in \Gamma$,
3) Γ^i is a Hilbert space for each $i \in \mathcal{M}$,
4) $\mathcal{J}(\boldsymbol{\gamma}) < \infty$ for all $\boldsymbol{\gamma} \in \Gamma$,

5) For all realizations ξ of Ξ and $u=(u^1,\dots,u^n)$ of U, we define $D_i\big(\xi,u^1,\dots,u^n\big) \coloneqq \nabla_{u^i} C\big(\xi,u^1,\dots,u^n\big)$. Then, for all $\boldsymbol{\gamma}\in\Gamma$,

$$\mathbb{E}[D_i\big(\Xi,\gamma^1(Z^1),\dots,\gamma^n(Z^n)\big)|Z^i] \in \Gamma^i \quad \forall i \in \mathcal{M}. \tag{5}$$

Then, if $\boldsymbol{\gamma}^*\in\Gamma$ is a stationary strategy, it is also optimal. □

Lemma 1. *Problem 1 satisfies all five aforementioned conditions for optimality of stationary strategies.*

Proof. See Section VI. □

V. Finding a Stationary Strategy

Lemma 1 implies that if we can find a stationary strategy, then it is guaranteed to be optimal. We consider linear control strategies of the form $\gamma^i(Z^i)=\Pi^i Z^i$ for all $i\in\mathcal{M}$. According to (4), this control strategy is stationary, if it satisfies the following equations for all realizations z^i of Z^i,

$$\nabla_{u^i}\,\mathbb{E}\left[C\big(\Xi,\{\Pi^j Z^j\}_{j\in\mathcal{M}\setminus\{i\}},u^i\big)|Z^i=z^i\right]\Big|_{u^i=\Pi^i z^i} = 0,\;\; i\in\mathcal{M}. \tag{6}$$

First, note that using (2), we have,

$$\begin{aligned} C\big(\Xi,\{\Pi^j Z^j\}_{j\in\mathcal{M}\setminus\{i\}},u^i\big) &= u^{i\intercal}R_{ii}u^i + 2u^{i\intercal}\sum_{j\in\mathcal{M}\setminus\{i\}} R_{ij}\Pi^j Z^j + \sum_{j,k\in\mathcal{M}\setminus\{i\}} Z^{j\intercal}\Pi^{j\intercal}R_{jk}\Pi^k Z^k + 2u^{i\intercal}S_{r,i}\Xi \\ &\quad + 2\sum_{j\in\mathcal{M}\setminus\{i\}} Z^{j\intercal}\Pi^{j\intercal}S_{r,j}\Xi + \Xi^\intercal Q\Xi. \end{aligned} \tag{7}$$

Then,

$$\begin{aligned} \mathbb{E}\left[C\big(\Xi,\{\Pi^j Z^j\}_{j\in\mathcal{M}\setminus\{i\}},u^i\big)|Z^i=z^i\right] &= u^{i\intercal}R_{ii}u^i + 2u^{i\intercal}\sum_{j\in\mathcal{M}\setminus\{i\}} R_{ij}\Pi^j\,\mathbb{E}[Z^j|Z^i=z^i] \\ &\quad + \sum_{j,k\in\mathcal{M}\setminus\{i\}} \mathbb{E}[Z^{j\intercal}\Pi^{j\intercal}R_{jk}\Pi^k Z^k|Z^i=z^i] \\ &\quad + 2u^{i\intercal}S_{r,i}\,\mathbb{E}[\Xi|Z^i=z^i] + 2\sum_{j\in\mathcal{M}\setminus\{i\}} \mathbb{E}[Z^{j\intercal}\Pi^{j\intercal}S_{r,j}\Xi|Z^i=z^i] \\ &\quad + \mathbb{E}[\Xi^\intercal Q\Xi|Z^i=z^i], \end{aligned} \tag{8}$$

$$\begin{aligned} \nabla_{u^i}\,\mathbb{E}\left[C\big(\Xi,\{\Pi^j Z^j\}_{j\in\mathcal{M}\setminus\{i\}},u^i\big)|Z^i=z^i\right]\Big|_{u^i=\Pi^i z^i} &= 2R_{ii}\Pi^i z^i + 2\sum_{j\in\mathcal{M}\setminus\{i\}} R_{ij}\Pi^j\,\mathbb{E}[Z^j|Z^i=z^i] \\ &\quad + 2S_{r,i}\,\mathbb{E}[\Xi|Z^i=z^i]. \end{aligned} \tag{9}$$

The vectors $\Xi, Z^i, i\in\mathcal{M}$, are jointly Gaussian. Consequently,

$$\begin{aligned} \mathbb{E}[Z^j|Z^i=z^i] &= \Sigma_{Z^jZ^i}\Sigma^{-1}_{Z^iZ^i}z^i \quad \forall j\in\mathcal{M}\setminus\{i\} \\ \mathbb{E}[\Xi|Z^i=z^i] &= \Sigma_{\Xi Z^i}\Sigma^{-1}_{Z^iZ^i}z^i \end{aligned} \tag{10}$$

where,

$$\Sigma_{Z^jZ^i} = \mathbb{E}[Z^j(Z^i)^\intercal] = \mathbb{E}[H^j\Xi\Xi^\intercal H^{i\intercal}] = H^j\Sigma H^{i\intercal} \quad \forall j \in \mathcal{M}\setminus\{i\},$$
$$\Sigma_{\Xi Z^i} = \mathbb{E}[\Xi Z^{i\intercal}] = \mathbb{E}[\Xi\Xi^\intercal H^{i\intercal}] = \Sigma H^{i\intercal}. \tag{11}$$

Considering (9) and the fact that $\mathbb{E}[Z^j|Z^i = z^i]$ and $E[\Xi|Z^i = z^i]$ are as calculated in (10), the conditions of (6) can be simplified as follows,

$$R_{ii}\Pi^i z^i + \sum_{j\in\mathcal{M}\setminus\{i\}} R_{ij}\Pi^j\Sigma_{Z^jZ^i}\Sigma^{-1}_{Z^iZ^i}z^i + S_{r,i}\Sigma_{\Xi Z^i}\Sigma^{-1}_{Z^iZ^i}z^i = 0 \qquad \forall i \in \mathcal{M}. \tag{12}$$

Since (12) should be true for all realization z^i of Z^i, we get,

$$\sum_{j=1}^{n} R_{ij}\Pi^j\Sigma_{Z^jZ^i}\Sigma^{-1}_{Z^iZ^i} = -S_{r,i}\Sigma_{\Xi Z^i}\Sigma^{-1}_{Z^iZ^i} \qquad \forall i \in \mathcal{M}. \tag{13}$$

We rewrite this result in the following Lemma.

Theorem 1. *The linear control strategy $\gamma^j(Z^j) = \Pi^j Z^j$ for all $j \in \mathcal{M}$ is team optimal for Problem 1 if the following equations have a solution for Π^j, $j \in \mathcal{M}$,*

$$\sum_{j=1}^{n} R_{ij}\Pi^j\Sigma_{Z^jZ^i} = -S_{r,i}\Sigma_{\Xi Z^i} \qquad \forall i \in \mathcal{M}. \tag{14}$$

Proof. By multiplying both sides of (13) from right side by $\Sigma_{Z^iZ^i}$, we can get (14). If (14) has a solution for Π^j, $j \in \mathcal{M}$, then the strategy $\gamma^j(Z^j) = \Pi^j Z^j$, $j \in \mathcal{M}$, satisfies the conditions for being a stationary strategy. Since we have shown that Problem 1 satisfies the sufficient conditions for optimality of stationary strategies, we can conclude that the strategy $\gamma^j(Z^j) = \Pi^j Z^j$, $j \in \mathcal{M}$, is team optimal. □

Next, we provide an example where the system of equations of (14) has a solution.

Example 1. *Consider a team problem with 2 members. The information available to team members is as follows,*

$$Z^1 = H^1\Xi = \begin{bmatrix}1 & 0\end{bmatrix}\Xi, \qquad Z^2 = H^2\Xi = \begin{bmatrix}1 & 1\end{bmatrix}\Xi$$

where $\Xi \sim \mathcal{N}(0, I)$. Hence,

$$\Sigma_{\Xi Z^1} = \begin{bmatrix}1\\0\end{bmatrix}, \quad \Sigma_{\Xi Z^2} = \begin{bmatrix}1\\1\end{bmatrix}, \quad \Sigma_{Z^1Z^1} = 1, \quad \Sigma_{Z^1Z^2} = \Sigma_{Z^2Z^1} = 1, \quad \Sigma_{Z^2Z^2} = 2.$$

Member $i = 1, 2$, chooses control action $U^i \in \mathbb{R}$ as a function of Z^i, that is, $U^i = \gamma^i(Z^i)$, where γ^i is the control strategy of member i. The performance of control actions U^1 and U^2 is measured by the following cost function

$$C(\Xi, U^1, U^2) = (M\Xi + U^1 + 2U^2)^2 = (\begin{bmatrix}1 & 1\end{bmatrix}\Xi + U^1 + 2U^2)^2$$
$$= \begin{bmatrix}U^1 & U^2\end{bmatrix} R \begin{bmatrix}U^1\\U^2\end{bmatrix} + 2\begin{bmatrix}U^1 & U^2\end{bmatrix} S\Xi + \Xi^\intercal \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\Xi$$

where $R = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ *and* $S = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$. *Note that* R *is not positive definite.*

According to Theorem 1, the linear control strategy $U^j = \Pi^j Z^j$, $j = 1, 2$, *is team optimal if the following equations have a solution for* Π^1 *and* Π^2,

$$\begin{aligned} \Pi^1 + 2\Pi^2 &= -1 \\ 2\Pi^1 + 8\Pi^2 &= -4. \end{aligned}$$

By solving the above system of equations, Π^1 *and* Π^2 *can be found to be*

$$\begin{bmatrix} \Pi^1 \\ \Pi^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5 \end{bmatrix}.$$

Therefore, the control strategies $U^1 = 0$ *and* $U^2 = -0.5Z^2$ *are team optimal.*

Remark 2. *The system of equations in* (14) *can be written as,*

$$\sum_{j=1}^{N} (\Sigma_{Z^iZ^j} \otimes R_{ij})(\Pi^j)_v = (-S_{r,i}\Sigma_{\Xi Z^i})_v \quad i = 1, 2, \dots, N$$

where $(\Pi^j)_v$ *denotes the vectorization of the matrix* Π^j *formed by stacking the columns of* Π^j *into a single column vector and* $\otimes$ *is the Kronecker product. Let's define* $G^{ij} = \Sigma_{Z^iZ^j} \otimes R_{ij}$, $Y^j = (\Pi^j)_v$ *and* $W^i = (-S_{r,i}\Sigma_{\Xi Z^i})_v$. *Note that* G^{ij} *is a* $d_{u^i}d_{z^i} \times d_{u^j}d_{z^j}$ *matrix and* Y^j *and* W^i *are column vectors with the size of* $d_{u^j}d_{z^j}$ *and* $d_{u^i}d_{z^i}$, *respectively. Then, we have*

$$\sum_{j=1}^{N} G^{ij}Y^j = W^i, \quad i = 1, 2, \dots, N,$$

which can be written as $GY = W$, *where*

$$G = \begin{bmatrix} G^{11} & \dots & G^{1N} \\ G^{21} & \dots & G^{2N} \\ \vdots & \ddots & \vdots \\ G^{N1} & \dots & G^{NN} \end{bmatrix}, \quad Y = \begin{bmatrix} Y^1 \\ Y^2 \\ \vdots \\ Y^N \end{bmatrix}, \quad W = \begin{bmatrix} W^1 \\ W^2 \\ \vdots \\ W^N \end{bmatrix}.$$

Note that Y *and* W *are column vectors both with the size of* $\sum_{i=1}^{N} d_{u^i}d_{z^i}$ *and* G *is a* $\sum_{i=1}^{N} d_{u^i}d_{z^i} \times \sum_{i=1}^{N} d_{u^i}d_{z^i}$ *square matrix. All solutions of* $GY = W$ *(if any exist) are given as* $Y = G^\dagger W + (I - G^\dagger G)T$, *where* T *is any arbitrary vector with the same size as vector* W *and* $G^\dagger$ *is the Moore-Penrose pseudo-inverse of* G. *A necessary and sufficient condition for any solution(s) to exist is that* $GG^\dagger W = W$.

In the case of Example 1,

$$G = \begin{bmatrix} \Sigma_{Z^1Z^1} \otimes R_{11} & \Sigma_{Z^1Z^2} \otimes R_{12} \\ \Sigma_{Z^2Z^1} \otimes R_{21} & \Sigma_{Z^2Z^2} \otimes R_{22} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix}, \quad Y = \begin{bmatrix} \Pi^1 \\ \Pi^2 \end{bmatrix}, \quad W = \begin{bmatrix} -S_{r,1}\Sigma_{\Xi Z^1} \\ -S_{r,2}\Sigma_{\Xi Z^2} \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}.$$

Since G is invertible, we have

$$G^{\dagger} = G^{-1} = \begin{bmatrix} 2 & -0.5 \\ -0.5 & 0.25 \end{bmatrix}.$$

Furthermore, since $GG^{\dagger}W = \begin{bmatrix} -1 \\ -4 \end{bmatrix} = W$, we know the solution(s) exist and are given by,

$$Y = \begin{bmatrix} \Pi^1 \\ \Pi^2 \end{bmatrix} = G^{\dagger}W + (I - G^{\dagger}G)T = \begin{bmatrix} 0 \\ -0.5 \end{bmatrix}$$

which is the same as our previous result.

Lemma 2. *Let an $m \times m$ matrix A be partitioned into the $m_i \times m_j$ blocks A^{ij} and $n \times n$ matrix B into the $n_k \times n_l$ blocks B^{kl} where $\sum_i m_i = m$, $\sum_j m_j = m$, $\sum_k n_k = n$, and $\sum_l n_l = n$.*

1) *The ij-th block of Khatri-Rao product of A and B, denoted by $A * B$, is defined as $(A * B)^{ij} = A^{ij} \otimes B^{ij}$ which is the $m_i n_i \times m_j n_j$ sized Kronecker product of A^{ij} and B^{ij} [4]. Then, the following equalities hold [5]:*
   a) $(A * B)^{\intercal}(A * B) = (A^{\intercal}A) \odot (B^{\intercal}B)$
   b) $(A * B)^{\dagger} = [(A^{\intercal}A) \odot (B^{\intercal}B)]^{\dagger}(A * B)^{\intercal}$

   where $\odot$ denotes the Hadamard product.
2) *If A is a symmetric positive definite matrix and B is a symmetric positive semi-definite matrix such that B^{kk} block matrix is positive definite for every k, then the Hadamard product of A and B is positive definite [1].*

Remark 3. *Let's denote the matrix composed of $\Sigma_{Z^iZ^j} = H^i\Sigma(H^j)^{\intercal}$ as its ij-th block by $\bar{\Sigma}$, then $G = \bar{\Sigma} * R$. According to Lemma 2, the Moore-Penrose pseudo-inverse of G can be found as $G^{\dagger} = [(\bar{\Sigma}^{\intercal}\bar{\Sigma}) \odot (R^{\intercal}R)]^{\dagger}(\bar{\Sigma} * R)^{\intercal}$.*

Remark 4. *Note that $\bar{\Sigma}$ is a symmetric positive semi-definite matrix because it can be written as*

$$\bar{\Sigma} = \begin{bmatrix} H^1 \\ H^2 \\ \vdots \\ H^N \end{bmatrix} \Sigma \begin{bmatrix} H^1 & H^2 & \dots & H^N \end{bmatrix}.$$

Furthermore, we assumed that $\bar{\Sigma}^{ii} = H^i\Sigma(H^i)^{\intercal}$ is positive definite for $i = 1, 2, \dots, N$. If R is a symmetric positive definite matrix [2], then according to Lemma 2, $(\bar{\Sigma}^{\intercal}\bar{\Sigma}) \odot (R^{\intercal}R)$ is positive definite and invertible. Furthermore, G is symmetric and according to Lemma 2, $G^{\intercal}G = G^2 = (\bar{\Sigma}^{\intercal}\bar{\Sigma}) \odot (R^{\intercal}R)$. Therefore, G^2 and consequently G are invertible, that is $G^{\dagger}G = GG^{\dagger} = I$. Then, according to Remark 2, $GY = W$ has the unique solution of $Y = G^{-1}W$. In order words, in this, there exists a unique linear stationary strategy.

VI. Proof of Lemma 1

We verify the five conditions for optimality of stationary strategies below:

1) In the cost function of (2), R is positive semi-definite because for any vector v, $v^\intercal R v = v^\intercal N^\intercal N v = (Nv)^\intercal (Nv) \geq 0$. Hence, $C(\xi, u)$ is convex in u.
   It is easy to verify that for all realizations u of U and ξ of Ξ, $\nabla_u C(\xi, u)$ exists and is continuous.
2) $\mathcal{J}(\boldsymbol{\gamma})$ is bounded from below for all $\boldsymbol{\gamma} \in \Gamma$ because for any realization ξ of Ξ,
$$C\big(\xi, \gamma^1(z^1), \ldots, \gamma^n(z^n)\big) = \Big(M\xi + \sum_{i=1}^{n} N_{c,i}\gamma^i(z^i)\Big)^\intercal \Big(M\xi + \sum_{i=1}^{n} N_{c,i}\gamma^i(z^i)\Big) \geq 0$$
   where $z^i = H^i\xi$ for all $i \in \mathcal{M}$. Hence, $\mathcal{J}(\boldsymbol{\gamma}) = \mathbb{E}[C\big(\Xi, \gamma^1(Z^1), \ldots, \gamma^n(Z^n)\big)] \geq 0$.

3) Note that Γ^i can be considered as the space of all Borel measurable functions $\gamma^i : \mathbb{R}^{d_{z^i}} \to \mathbb{R}^{d_{u^i}}$, defined on the probability space $\{\mathbb{R}^{d_{z^i}}, \mathscr{B}(\mathbb{R}^{d_{z^i}}), P\}$[1], which have finite second moments, that is, $\mathbb{E}[\gamma^i(Z^i)^\intercal \gamma^i(Z^i)] < \infty$. Let $\gamma^i = vec(\gamma^i_1, \gamma^i_2, \ldots, \gamma^i_{d_{u^i}})$ where $\gamma^i_l : \mathbb{R}^{d_{z^i}} \to \mathbb{R}$ for $l = 1, 2, \ldots, d_{u^i}$. Then, the Borel measurability of γ^i_l for $l = 1, 2, \ldots, d_{u^i}$ results from the Borel measurability of γ^i. Furthermore, if function γ^i has a finite second moment, functions γ^i_l for $l = 1, 2, \ldots, d_{u^i}$ have finite second moments, that is $\mathbb{E}[\big(\gamma^i_l(Z^i)\big)^2] < \infty$.
   Denote by Γ^i_l the space of all Borel measurable functions $\gamma^i_l : \mathbb{R}^{d_{z^i}} \to \mathbb{R}$ defined on the probability space $\{\mathbb{R}^{d_{z^i}}, \mathscr{B}(\mathbb{R}^{d_{z^i}}), P\}$ and with the finite second moments. Then $\Gamma^i = \prod_{l=1}^{d_{u^i}} \Gamma^i_l$. Now consider the space Γ^i_l endowed with the inner product $\langle f_l, g_l \rangle = \mathbb{E}[f_l(Z^i) g_l(Z^i)]$, associated norm $\|f_l\| = (\langle f_l, f_l \rangle)^{1/2}$, and metric $\|f_l - g_l\|$ where $f_l, g_l \in \Gamma^i_l$. Then, Γ^i_l is a Hilbert space for $l = 1, 2, \ldots, d_{u^i}$ [6]. For $f, g \in \Gamma^i$, let's define $\langle f, g \rangle := \mathbb{E}[f(Z^i)^\intercal g(Z^i)]$ be a mapping form $\Gamma^i \times \Gamma^i$ to $\mathbb{R}$. Then,
$$\begin{aligned}\langle f, g \rangle &= \mathbb{E}[f(Z^i)^\intercal g(Z^i)] = \mathbb{E}[vec\big(f_1(Z^i), f_2(Z^i), \ldots, f_{d_{u^i}}(Z^i)\big)^\intercal vec\big(g_1(Z^i), g_2(Z^i), \ldots, g_{d_{u^i}}(Z^i)\big)] \\ &= \mathbb{E}[\sum_{l=1}^{d_{u^i}} f_l(Z^i) g_l(Z^i)] = \sum_{l=1}^{d_{u^i}} \mathbb{E}[f_l(Z^i) g_l(Z^i)] = \sum_{l=1}^{d_{u^i}} \langle f_l, g_l \rangle. \qquad (15)\end{aligned}$$
   Since $\langle f_l, g_l \rangle$ for $l = 1, 2, \ldots, d_{u^i}$ is an inner product, we can easily conclude that $\langle f, g \rangle$ is an inner product. Hence, Γ^i is a vector space equipped with $\langle f, g \rangle = \mathbb{E}[f(Z^i)^\intercal g(Z^i)]$ as an inner product and $\|f\| = (\langle f, f \rangle)^{1/2}$ as the corresponding norm. Furthermore, for $f, g \in \Gamma^i$, we can define the metric (distance) of $\|f - g\|$. Therefore, Γ^i is a pre-Hilbert space.
   In order to be a Hilbert space, we need the space Γ^i to be complete. That is, we need to show that every Cauchy sequence in Γ^i has a limit point that belongs to Γ^i. To show this, let $\{f^n\} \subset \Gamma^i$ be a Cauchy sequence, that is $\|f^m - f^n\| \to 0$ as $m, n \to \infty$. This implies that $\{f^n_l\}$ is a Cauchy sequence in Γ^i_l for $l = 1, 2, \ldots, d_{u^i}$ because,
$$\|f^m - f^n\|^2 = \langle f^m - f^n, f^m - f^n \rangle = \sum_{l=1}^{d_{u^i}} \langle f^m_l - f^n_l, f^m_l - f^n_l \rangle = \sum_{l=1}^{d_{u^i}} \|f^m_l - f^n_l\|^2$$
   and
$$\|f^m - f^n\| \to 0 \text{ as } m, n \to \infty \Longrightarrow \|f^m_l - f^n_l\| \to 0 \text{ as } m, n \to \infty \quad \forall l = 1, 2, \ldots, d_{u^i}.$$

[1] P is the probability measure induced by the Gaussian density of Z^i.

Since Γ_l^i is a Hilbert space, the Cauchy sequence $\{f_l^n\}$ converges to a limit point $f_l \in \Gamma_l^i$ (that is $\|f_l^n - f_l\| \to 0 \text{ as } n \to \infty$) for $l = 1, 2, \ldots, d_{u^i}$. If we define $f = \mathit{vec}(f_1, f_2, \ldots, f_{d_{u^i}})$, then $f \in \Gamma^i$ because according to (15)

$$\langle f, f \rangle = \sum_{l=1}^{d_{u^i}} \langle f_l, f_l \rangle < \infty.$$

Furthermore, $\{f^n\}$ converges to f because

$$\|f^n - f\|^2 = \langle f^n - f, f^n - f \rangle = \sum_{l=1}^{d_{u^i}} \langle f_l^n - f_l, f_l^n - f_l \rangle = \sum_{l=1}^{d_{u^i}} \|f_l^n - f_l\|^2$$

and

$$\|f_l^n - f_l\| \to 0 \text{ as } n \to \infty \quad \forall l = 1, 2, \ldots, d_{u^i}.$$

Therefore, Γ^i is a Hilbert space.

4) To show that $\mathcal{J}(\boldsymbol{\gamma}) < \infty$ for all $\boldsymbol{\gamma} \in \Gamma$, note that $\mathcal{J}(\boldsymbol{\gamma})$ can be written as follows,

$$\mathcal{J}(\boldsymbol{\gamma}) = \mathbb{E}[C(\Xi, \boldsymbol{\gamma})] = \mathbb{E}\Big[\begin{bmatrix} \gamma^1(Z^1)^\intercal & \gamma^2(Z^2)^\intercal & \ldots & \gamma^n(Z^n)^\intercal \end{bmatrix} R \begin{bmatrix} \gamma^1(Z^1) \\ \gamma^2(Z^2) \\ \vdots \\ \gamma^n(Z^n) \end{bmatrix} \Big]$$
$$+ 2\,\mathbb{E}[\sum_{i=1}^{n} \gamma^i(Z^i)^\intercal S_{r,i} \Xi] + \mathbb{E}[\Xi^\intercal Q \Xi]. \tag{16}$$

Lemma 3. *Assume $X \in \mathbb{R}^n$ is a random vector and Q is a deterministic, symmetric positive semi-definite $n \times n$ matrix. Then, there is a non-negative constant α such that $\mathbb{E}[X^\intercal Q X] \leq \alpha\, \mathbb{E}\left[X^\intercal X\right]$.*

Proof. For any vector x, $x^\intercal Q x \leq \lambda_{max} x^\intercal x$, where $\lambda_{max} \geq 0$ is the largest eigenvalue of Q. Taking expectation of the above inequality proves the lemma. $\square$

Let's define $\boldsymbol{\gamma}(Z) := \mathit{vec}\big(\gamma^1(Z^1), \gamma^2(Z^2), \ldots, \gamma^n(Z^n)\big)$, then the first term of (16) can be written as $\mathbb{E}[\boldsymbol{\gamma}(Z)^\intercal R \boldsymbol{\gamma}(Z)]$. Since R is symmetric positive semi-definite, Lemma 3 implies that for some $\alpha \geq 0$,

$$\mathbb{E}[\boldsymbol{\gamma}(Z)^\intercal R \boldsymbol{\gamma}(Z)] \leq \alpha\, \mathbb{E}\left[\boldsymbol{\gamma}(Z)^\intercal \boldsymbol{\gamma}(Z)\right] = \alpha \sum_{i=1}^{n} \mathbb{E}[\gamma^i(Z^i)^\intercal \gamma^i(Z^i)] < \infty, \tag{17}$$

where the last inequality is true because $\mathbb{E}[\gamma^i(Z^i)^\intercal \gamma^i(Z^i)] < \infty$ for all i.
For the second term of (16),

$$\Big| \mathbb{E}\,[\sum_{i=1}^{n} \gamma^i(Z^i)^\intercal S_{r,i} \Xi] \Big| = \Big| \sum_{i=1}^{n} \mathbb{E}[\gamma^i(Z^i)^\intercal S_{r,i} \Xi] \Big| \leq \sum_{i=1}^{n} \big(\mathbb{E}[\gamma^i(Z^i)^\intercal \gamma^i(Z^i)] \big)^{1/2} \big(\mathbb{E}[(S_{r,i}\Xi)^\intercal S_{r,i} \Xi] \big)^{1/2}$$
$$= \sum_{i=1}^{n} \big(\mathbb{E}[\gamma^i(Z^i)^\intercal \gamma^i(Z^i)] \big)^{1/2} \big(\mathbb{E}[\Xi^\intercal S_{r,i}^\intercal S_{r,i} \Xi] \big)^{1/2} < \infty, \tag{18}$$

where the first inequality is true because of *Cauchy-Schwarz inequality* on the Hilbert space of all Borel measurable functions from $\mathbb{R}^{d_\xi}$ to $\mathbb{R}^{d_{u^i}}$, defined on the probability space $\{\mathbb{R}^{d_\xi}, \mathscr{B}(\mathbb{R}^{d_\xi}), P\}$[2], which have finite second moments. The second inequality is true because according to Lemma 3,

$$\mathbb{E}[\Xi^\intercal S_{r,i}^\intercal S_{r,i}\Xi] \leq \alpha\,\mathbb{E}[\Xi^\intercal \Xi] = \alpha\,\mathbb{E}[\mathbf{tr}(\Xi^\intercal\Xi)] = \alpha\,\mathbb{E}[\mathbf{tr}(\Xi\Xi^\intercal)] = \alpha\,\mathbf{tr}(\mathbb{E}[\Xi\Xi^\intercal]) = \alpha\,\mathbf{tr}(\Sigma) < \infty. \tag{19}$$

Similarly,using Lemma 3 and (19), the last term of (16), that is $\mathbb{E}[\Xi^\intercal Q\Xi]$, can also be shown to be finite. Therefore, $\mathcal{J}(\boldsymbol{\gamma}) < \infty$ for all $\boldsymbol{\gamma} \in \Gamma$.

5) To show that (5) holds, according to (2), for realizations ξ and u of Ξ and U, $D_i(\xi, u)$ is as follows,

$$D_i(\xi, u) = 2R_{ii}u^i + 2\sum_{j\in\mathcal{M}\setminus\{i\}} R_{ij}u^j + 2S_{r,i}\xi. \tag{20}$$

By considering control strategies $U^i = \gamma^i(Z^i)$ for all $i \in \mathcal{M}$,

$$\mathbb{E}[D_i\big(\Xi, \gamma^1(Z^1), \dots, \gamma^n(Z^n)\big)|Z^i] = 2R_{ii}\gamma^i(Z^i) + 2\sum_{j\in\mathcal{M}\setminus\{i\}} R_{ij}\,\mathbb{E}[\gamma^j(Z^j)|Z^i] + 2S_{r,i}\,\mathbb{E}[\Xi|Z^i]. \tag{21}$$

To show that (21) belongs to Γ^i, it suffices to show that each term in the right hand side of (21) belongs to Γ^i (since Γ^i is a Hilbert space). Each of these terms is a mapping from $\mathbb{R}^{d_{z^i}} \to \mathbb{R}^{d_{u^i}}$, to show that each term belongs to Γ^i, we need to show that it has a finite second moment.

The first term $2R_{ii}\gamma^i(Z^i) \in \Gamma^i$ because according to Lemma 3,

$$\mathbb{E}[\gamma^i(Z^i)^\intercal R_{ii}^\intercal R_{ii}\gamma^i(Z^i)] \leq \alpha\,\mathbb{E}[\gamma^i(Z^i)^\intercal\gamma^i(Z^i)] < \infty, \tag{22}$$

where the last inequality is true because $\gamma^i \in \Gamma^i$.

The second term in the right hand side of (21) is a linear combination of terms $2R_{ij}\,\mathbb{E}[\gamma^j(Z^j)|Z^i]$ for $j \in \mathcal{M}\setminus\{i\}$ where each term belongs to Γ^i because,

$$\begin{aligned}&\mathbb{E}\left[\mathbb{E}[\gamma^j(Z^j)|Z^i]^\intercal R_{ij}^\intercal R_{ij}\,\mathbb{E}[\gamma^j(Z^j)|Z^i]\right] \leq \alpha\,\mathbb{E}\left[\mathbb{E}[\gamma^j(Z^j)|Z^i]^\intercal\,\mathbb{E}[\gamma^j(Z^j)|Z^i]\right]\\ &\leq \alpha\,\mathbb{E}\left[\mathbb{E}\left[\gamma^j(Z^j)^\intercal\gamma^j(Z^j)|Z^i\right]\right] = \alpha\,\mathbb{E}\left[\gamma^j(Z^j)^\intercal\gamma^j(Z^j)\right] < \infty,\end{aligned} \tag{23}$$

where the first inequality is true because of Lemma 3, the second inequality holds because of *Jensen's inequality*[3], and the last inequality is true because $\gamma^j \in \Gamma^j$. Furthermore, the first equality is true because of *smoothing property*[4]. Therefore, the second term in the right hand side of (21) belongs to Γ^i.

The last term in the right hand side of (21) also belongs to Γ^i because,

$$\begin{aligned}&\mathbb{E}\left[\mathbb{E}\left[\Xi|Z^i\right]^\intercal S_{r,i}^\intercal S_{r,i}\,\mathbb{E}\left[\Xi|Z^i\right]\right] \leq \alpha\,\mathbb{E}\left[\mathbb{E}\left[\Xi|Z^i\right]^\intercal\,\mathbb{E}\left[\Xi|Z^i\right]\right] \leq \alpha\,\mathbb{E}\left[\mathbb{E}\left[\Xi^\intercal\Xi|Z^i\right]\right]\\ &= \alpha\,\mathbb{E}\left[\Xi^\intercal\Xi\right] = \alpha\,\mathbf{tr}(\Sigma) < \infty,\end{aligned} \tag{24}$$

[2] P is the probability measure induced by the Gaussian density of Ξ.

[3] Jensen's inequality states: If $\psi : \mathbb{R} \to \mathbb{R}$ is convex and bounded from below, and if X is a random variable defined on the probability space $(\Omega, \mathcal{F}, P)$, then $\mathbb{E}[\psi(X)|\mathcal{H}] \geq \psi(\mathbb{E}[X|\mathcal{H}])$ where $\mathcal{H}$ is any sub σ-algebra of $\mathcal{F}$.

[4] According to smoothing property: If X is a random variable defined on the probability space $(\Omega, \mathcal{F}, P)$ and if $\mathcal{H}$ is any sub σ-algebra of $\mathcal{F}$, then $\mathbb{E}[X] = \mathbb{E}\left[\mathbb{E}[X|\mathcal{H}]\right]$.

where the first and the second inequalities are true because of Lemma 3 and the *Jensen's inequality*, respectively. Furthermore, the first equality is true because of the *smoothing property*.

Therefore, (21) belongs to Γ^i.

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